

# KNOTS WITH UNKNOTTING NUMBER ONE AND HEEGAARD FLOER HOMOLOGY

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**ABSTRACT.** We use Heegaard Floer homology to give obstructions to unknotting a knot with a single crossing change. These restrictions are particularly useful in the case where the knot in question is alternating. As an example, we use them to classify all knots with crossing number less than or equal to nine and unknotting number equal to one. We also classify alternating knots with ten crossings and unknotting number equal to one.

## 1. INTRODUCTION

The *unknotting number*  $u(K)$  of a knot  $K \subset S^3$  is the minimal number of crossing changes required to unknot  $K$ . One lower bound on this number is given by a result of Murasugi [16],

$$|\sigma(K)| \leq 2u(K),$$

where here  $\sigma(K)$  denotes the signature of the knot.

In this article, we focus on obstructions to a knot  $K$  having  $u(K) = 1$ . One classical observation is that if  $K$  has  $u(K) = 1$ , then the branched double-cover of  $S^3$  along  $K$ ,  $\Sigma(K)$ , has cyclic first homology, c.f. [17]. Another obstruction stems from the linking form of  $\Sigma(K)$ , c.f. [13]. Indeed, a number of other obstructions to  $K$  having  $u(K) = 1$  can be given by considering its branched double-cover. For example Kanenobu and Murakami in [7] use this construction, together with the cyclic surgery theorem of Culler, Gordon, Luecke, and Shalen [4] to classify all two-bridge knots with  $u(K) = 1$ . More recently, Gordon and Luecke [6] study this problem for knots whose branched double-cover is toroidal. The aim of the present article is to give a new obstruction for a knot to have unknotting number equal to one, which uses Heegaard Floer homology [28] for the branched double-cover.

The obstruction is most easily stated in the case where  $K$  is an alternating knot. Indeed, in this case, a statement can be given purely in terms of elementary number-theoretic properties of the Goeritz matrix of  $K$ . We give some necessary background.

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*Key words and phrases.* alternating knots, unknotting number one, Floer homology, Goeritz matrix.

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Let

$$Q: V \otimes V \longrightarrow \mathbb{Z}$$

be a negative-definite quadratic form over a lattice  $V$ .  $Q$  determines a map

$$q: V \longrightarrow V^*,$$

where here  $V^*$  denotes  $\text{Hom}(V, \mathbb{Z})$ , and it induces a bilinear form with values in the rationals

$$Q^*: V^* \otimes V^* \longrightarrow \mathbb{Q}.$$

We suppose that the number of elements in the cokernel of  $q$  is odd. A dual element  $K \in V^*$  is called *characteristic* if

$$\langle K, v \rangle \equiv Q(v \otimes v) \pmod{2}$$

for all  $v \in V$ . Fix an element  $v_0 \in V$  with the property that  $q(v_0)$  is characteristic. We now define a map

$$M_Q: \text{Coker}(q) \longrightarrow \mathbb{Q}$$

by

$$4M_Q(\xi) = \max_{\{v \in V^* \mid [v] = \xi\}} Q^*\left((q(v_0) + 2v) \otimes (q(v_0) + 2v)\right) + \text{rk}(V).$$

Since  $Q$  is negative-definite, the maximum exists; and since the number of elements in the cokernel of  $q$  is odd, the function is independent of the choice of  $v_0$ .

Let  $K$  be an alternating knot. The projection can be used to give a planar graph  $G$ , whose vertices correspond to the white regions in the checkerboard coloring of the knot projection. In fact, by choosing a regular alternating projection, we can assume that in the white graph, there are no edges connecting a vertex to itself. Let  $V$  denote the integral lattice formally generated by the vertices, modulo the relation that  $\sum_{v \in V} v = 0$ . Recall from [3] that the Goeritz form corresponding to this projection of  $K$  is the quadratic form

$$Q: V \otimes V \longrightarrow \mathbb{Z}$$

defined by the rule that if  $v$  and  $w$  are distinct vertices of  $G$ , then  $Q(v \otimes w)$  is the number of edges connecting  $v$  to  $w$ , and also  $Q(v \otimes v) = -\deg(v)$ , where here  $\deg(v)$  denotes the number of edges containing  $v$ . The number of elements in the cokernel of the associated linear map  $q$  is the determinant  $D$  of the knot  $K$ .

Consider the quadratic form  $R_{2n-1}$  on  $\mathbb{Z} \oplus \mathbb{Z}$  represented by the matrix

$$\begin{pmatrix} -n & 1 \\ 1 & -2 \end{pmatrix}.$$

We define

$$\phi_0: \mathbb{Z}/(2n-1)\mathbb{Z} \xrightarrow{\cong} \text{Coker } R_{2n-1}$$

by the property that  $\phi_0(i) \in V^*$  is the homomorphism whose value on  $(1, 0)$  is  $i$  and on  $(0, 1)$  is 0. Correspondingly we define a map  $\gamma_{2n-1}: \mathbb{Z}/(2n-1)\mathbb{Z} \rightarrow \mathbb{Q}$  by

$$\gamma_{2n-1}(i) = M_{R_{2n-1}}(\phi_0(i)).$$

Given an alternating projection of a knot  $K$  with determinant  $D$ , Goeritz form  $Q$ , an isomorphism  $\phi: \mathbb{Z}/D\mathbb{Z} \rightarrow \text{Coker}(q)$ , and a sign  $\epsilon = \pm 1$ , we define a function  $T_{\phi, \epsilon}: \mathbb{Z}/D\mathbb{Z} \rightarrow \mathbb{Q}$  by

$$T_{\phi, \epsilon}(i) = -\epsilon \cdot M_Q(\phi(i)) - \gamma_D(i).$$

Note that such a map satisfies the symmetry  $T_{\phi, \epsilon}(i) = T_{\phi, \epsilon}(-i)$ .

**Theorem 1.1.** *Let  $K$  be an alternating knot with determinant  $D$ , and let  $Q$  be the negative-definite Goeritz form corresponding to a regular, alternating projection of  $K$ . If  $u(K) = 1$ , then there must be an isomorphism  $\phi: \mathbb{Z}/D\mathbb{Z} \rightarrow \text{Coker}(q)$  and a sign  $\epsilon \in \{\pm 1\}$  with the properties that for all  $i \in \mathbb{Z}/D\mathbb{Z}$ :*

- (1)  $T_{\phi, \epsilon}(i) \equiv 0 \pmod{2}$
- (2)  $T_{\phi, \epsilon}(i) \geq 0$ .

Moreover, if  $K$  is an alternating knot with  $u(K) = 1$ , and furthermore  $|M_{Q_K}(0)| \leq \frac{1}{2}$ , then there is a choice of  $\epsilon$  and  $\phi$  satisfying Equations (1) and (2), and the following additional symmetry:

- (3)  $T_{\phi, \epsilon}(i) = T_{\phi, \epsilon}(2k - i)$

for  $1 \leq i < k$  when  $D = 4k - 1$  and for  $0 \leq i < k$  when  $D = 4k + 1$ .

The obstruction given in Theorem 1.1 does not depend on the choice of alternating projection of  $K$  (c.f. Section 5). Note also that there are stronger versions of the above result, c.f. Theorem 8.1 and 8.4.

As an application of the above obstruction, we study knots with small ( $\leq 10$ ) crossing number. Note that if a non-prime knot has crossing number  $\leq 10$ , then  $H_1(\Sigma(K); \mathbb{Z})$  is non-cyclic, and hence it does not have  $u(K) = 1$ . Alternatively, one can appeal to a general result of Scharlemann [33], according to which knots with  $u(K) = 1$  are prime. So, we will always restrict our attention to the prime case here.

We begin with the case of knots with nine or fewer crossings. Among these, three were listed in Kawauchi's table as having unknown  $u = 1$  or 2. For these, we get the following result:

**Corollary 1.2.** *The knots  $8_{10}$ ,  $9_{29}$ , and  $9_{32}$  all have unknotting number equal to two.*

In recent work [6], Gordon and Luecke have also shown that  $9_{29}$  and  $9_{32}$  have  $u = 2$ .

Among knots which admit a diagram with nine or fewer crossings, the signature, cyclicity of  $H_1(\Sigma(K))$ , and Theorem 1.1 are sufficient to determine all of those with  $u = 1$ . We review here some earlier results in this direction. In this family there are 58 knots with  $|\sigma| \leq 2$ , c.f. [8] see also [9]. Of these knots, 35 can be directly seen to

have  $u = 1$ . Of the remaining 23, eight ( $8_{18}$ ,  $9_{35}$ ,  $9_{37}$ ,  $9_{40}$ ,  $9_{41}$ ,  $9_{46}$ ,  $9_{47}$ , and  $9_{48}$ ) are ruled out by the fact that  $H_1(\Sigma(K); \mathbb{Z})$  is non-cyclic, see also [17]. Of the remaining 15, one ( $7_4$ ) is ruled out by Lickorish [13], nine ( $8_3$ ,  $8_6$ ,  $8_8$ ,  $8_{12}$ ,  $9_5$ ,  $9_8$ ,  $9_{15}$ ,  $9_{17}$ , and  $9_{31}$ ) are ruled out Kanenobu and Murakami [7], one ( $9_{25}$ ) is ruled out by Kobayashi [10], and one more ( $8_{16}$ ) has been ruled out by Rickard, see also [15], [34]. This left open the unknotting status of  $8_{10}$ ,  $9_{32}$ , and  $9_{29}$  (note that  $9_{29}$  was mistakenly listed in some tables as having  $u = 1$ ). While Corollary 1.2 concerns these last three knots, same method (Theorem 1.1) also proves that the earlier 12 knots do not have  $u = 1$ .

We also consider knots with ten crossings<sup>1</sup>. There are many knots in Kawauchi's table [8] with unknown unknotting status. We considered the knots in the table which were listed as having unknotting number undetermined but possibly one. There are 40 such knots, of which 28 are alternating and 12 are not. Theorem 1.1 shows that none of these 28 alternating knots has  $u(K) = 1$ :

**Corollary 1.3.** *The following twenty-four ten-crossing alternating knots have unknotting number equal to two:*

$$\begin{array}{cccccccccccc} 10_{48} & 10_{52} & 10_{57} & 10_{58} & 10_{64} & 10_{67} & 10_{68} & 10_{70} & 10_{81} & 10_{83} & 10_{86} & 10_{87} \\ 10_{90} & 10_{93} & 10_{94} & 10_{96} & 10_{105} & 10_{106} & 10_{109} & 10_{110} & 10_{112} & 10_{116} & 10_{117} & 10_{121}; \end{array}$$

moreover, the knots

$$10_{51} \quad 10_{54} \quad 10_{77} \quad 10_{79}$$

have unknotting number equal to two or three.

Note that in an earlier preprint using the linking form, Stoimenow [34] has shown that the knots

$$10_{86}, 10_{105}, 10_{106}, 10_{109}, 10_{116}, 10_{121}.$$

have  $u = 2$ . In recent work, Gordon and Luecke [6] have shown the same results for

$$10_{79}, 10_{81}, 10_{83}, 10_{86}, 10_{87}, 10_{90}, 10_{93}, 10_{94}, 10_{96}.$$

Theorem 1.1, and the classical invariants (signature,  $H_1(\Sigma(K); \mathbb{Z})$ ) suffice to classify all 10-crossing alternating knots with  $u(K) = 1$ , see also Section 6.

Of the remaining twelve non-alternating ten-crossing knots in Kawauchi's table with unknown unknotting number equal possibly to one, one ( $10_{145}$ ) has been shown to have  $u = 2$  by Tanaka [35]. Note that  $u(10_{131}) = 1$  (c.f. [34], see also Figure 9 below). We resolve here the status of an additional 9 of these, using the methods for the proof of Theorem 1.1, which apply for certain non-alternating cases, c.f. Section 7 below.

**Corollary 1.4.** *The following nine non-alternating, ten-crossing knots*

$$10_{125} \quad 10_{126} \quad 10_{130} \quad 10_{135} \quad 10_{138} \quad 10_{148} \quad 10_{151} \quad 10_{158} \quad 10_{162}$$

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<sup>1</sup>In this paper, we use the numbering scheme on knots from Rolfsen's table [31], modified so that the Perko pair is removed. In particular,  $10_{161} \neq 10_{162}$ , and the pair  $10_{83}$  and  $10_{86}$  have been switched by comparison with Kawauchi's table.

have unknotting number equal to two.

Recent work of Gordon and Lucke [6] shows that

$$10_{148}, \quad 10_{151}, \quad 10_{153}$$

have  $u = 2$ . Thus, the results of this paper together with [6] completes the classification of all 10-crossing knots with  $u = 1$ .

**1.1. The basic idea.** We discuss some of the ingredients which go into the proof of Theorem 1.1. We begin with the following observation of Montesinos c.f. [14], [13], [2]:

**Lemma 1.5.** (*Montesinos*) *If  $K$  has  $u(K) = 1$ , then  $\Sigma(K) \cong S^3_{\pm D/2}(C)$  for some other knot  $C \subset S^3$ , where here  $D$  is the determinant of  $K$ .*

In using the above lemma, it is helpful to have an invariant for three-manifolds which one can calculate for a three-manifold given as a branched double-cover of  $S^3$  branched along a specific knot, and which can also detect obstructions to realizing a given three-manifold  $Y$  as  $n/2$  surgery on a knot in  $S^3$  (for integral  $n$ ). Such an obstruction is furnished by Heegaard Floer homology.

The algebraic structure of Heegaard Floer homology, together with an induced grading (which takes values in the rational numbers  $\mathbb{Q}$ ) gives rise to a function

$$d: H^2(\Sigma(K); \mathbb{Z}) \longrightarrow \mathbb{Q},$$

the “correction terms” for  $\Sigma(K)$  (c.f. [22], see also the discussion in Section 2 below). (Indeed, this map can be given for an arbitrary oriented rational homology-three-sphere  $Y$  instead of  $\Sigma(K)$ , except that in the general case, the correction terms should be interpreted as a rational-valued function on the  $\text{Spin}^c$  structures over  $Y$ .) The correction terms constrain the intersection form of any smooth four-manifold which bounds  $\Sigma(K)$ , according to Theorem 9.6 of [22] (restated in Theorem 2.1 below), which is analogous to a gauge-theoretic result of Frøyshov [5].

This fact, together with Lemma 1.5, leads at once to an obstruction to a knot having unknotting number one, stated in terms of the correction terms of  $\Sigma(K)$ , c.f. Theorem 3.1 below. Note that that result does not use the hypothesis that  $K$  is alternating.

In general, calculating Heegaard Floer homology, and even the correction terms  $d(Y, \mathfrak{s})$  for an arbitrary three-manifold can be quite challenging. However, this problem is easily solved for the branched double-covers of alternating knots, and in particular the correction terms correspond to the quantities  $M_Q$  for the Goeritz form (see Proposition 3.3 and Theorem 3.4 both in [25], restated as Proposition 3.2 below).

In fact, the calculation of the correction terms for alternating knots, together with the intersection form bounds are not sufficient for establishing the full statement of Theorem 1.1: they establish only Conditions (1) and (2). For the additional symmetry from Condition (3), we need to go further into the structure of the Floer homology of branched double-covers of alternating knots. Specifically, these are rational homology

three-spheres whose Heegaard Floer homology is as simple as possible: they are  $L$ -spaces in the sense of [20] and also Definition 2.3 below. To obtain the full statement of Theorem 1.1, we establish constraints on the correction terms of  $L$ -spaces which can be obtained as  $n/2$  surgery on a knot in  $S^3$ , c.f. Theorem 4.1 below.

**1.2. Comparison with other techniques.** There are other applications of gauge theory and Floer theory to studying the unknotting number of knots, c.f. [11], [32], [23], [30], [18]. These techniques all give various lower bounds on the four-ball genus, and hence the unknotting number. By contrast, the obstructions in this paper give information which is independent of the four-ball genus, and in particular they also give non-trivial bounds in cases where the four-ball genus is known to be zero or one.

**1.3. Logical dependence.** The proof of Theorem 1.1 uses Heegaard Floer homology, as introduced in [28]. The results here make heavy use of the surgery long exact sequence for Heegaard Floer homology, c.f. [27], and also properties of the rational grading on Floer homology, c.f. [22] (and its interaction with the long exact sequence). In addition, we do refer some to [25], a paper concerned with the Heegaard Floer homology of branched double-covers of knots, and specifically to the results when  $K$  is alternating. But these results are confined Section 3 of [25], and are all fairly straightforward consequences of the surgery long exact sequence and the rational gradings. Utilizing in addition some results from [20], we obtain stronger constraints on alternating knots with unknotting number one, as explained in Theorem 8.4 below. These stronger results are not needed, though, for the classification of alternating knots with unknotting number one with ten or fewer crossings.

**1.4. Organization.** The rest of this paper is organized as follows. In Section 2 we recall some of the essentials of the Heegaard Floer homology package which we use here, and specifically the constraints on intersection forms coming from the correction terms. In Section 3, we show how this leads quickly to an obstruction for an arbitrary knot having  $u(K) = 1$ , stated in terms of the correction terms of its branched double-cover (c.f. Theorem 3.1 below). In that section, we also recall the Heegaard Floer homology of branched double-covers of alternating knots, including the interpretation of their correction terms in terms of the Goeritz matrix. In Section 4, we prove a result about the correction terms of an  $L$ -space which can be obtained as  $D/2$  surgery on a knot in  $S^3$  (for some integer  $D$ ). This provides a key ingredient in establishing Condition (3) from Theorem 1.1. Theorem 1.1 is then proved in Section 5. In Section 6 we discuss the calculations which lead to the proofs of Corollaries 1.2 and 1.3. In Section 7, we turn our attention to some non-alternating knots, proving Corollary 1.4 above. In Section 8 we give refinements of Theorem 1.1. In particular, we show how the methods shed light on the problem of signed crossing (which we illustrate with the knot  $9_{33}$ ). Moreover, in that section we give an interpretation of the non-negative integers  $T_{\phi,\epsilon}$ .

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## 2. HEEGAARD FLOER HOMOLOGY

We give here a rapid outline of the Heegaard Floer homology needed in the present article. We consider oriented three-manifolds  $Y$  which are rational homology three-spheres (i.e. closed three-manifolds with  $H_1(Y; \mathbb{Q}) = 0$ ), and for simplicity, we use here Heegaard Floer homology with coefficients in a field  $\mathbb{F}$ , which we take to be  $\mathbb{Z}/2\mathbb{Z}$  for definiteness.

**2.1. Heegaard Floer homology for rational homology three-spheres and its  $\mathbb{Q}$ -grading.** Recall that if  $X$  is an oriented three- or four-manifold, the space of  $\text{Spin}^c$  structures  $\text{Spin}^c(X)$  is an affine space for  $H^2(X; \mathbb{Z})$ . Moreover, each  $\text{Spin}^c$  structure has a first Chern class in  $H^2(X; \mathbb{Z})$ , which is related to the action by the formula  $c_1(\mathfrak{s} + h) = c_1(\mathfrak{s}) + 2h$  for any  $h \in H^2(X; \mathbb{Z})$ .

When  $Y$  is an oriented rational homology three-sphere and  $\mathfrak{s}$  is a  $\text{Spin}^c$  structure over  $Y$ , its Heegaard Floer homology  $HF^+(Y, \mathfrak{s})$  is a  $\mathbb{Q}$ -graded module over the polynomial algebra  $\mathbb{F}[U]$ ,

$$HF^+(Y, \mathfrak{s}) = \bigoplus_{d \in \mathbb{Q}} HF_d^+(Y, \mathfrak{s}),$$

where multiplication by  $U$  lowers degree by two. In each grading,  $i \in \mathbb{Q}$ ,  $HF_i^+(Y, \mathfrak{s})$  is a finite-dimensional  $\mathbb{F}$ -vector space.

Indeed, there is another simpler variant of Heegaard Floer homology,  $HF^\infty(Y)$ , for which

$$(4) \quad HF^\infty(Y, \mathfrak{s}) \cong \mathbb{F}[U, U^{-1}]$$

for each  $\mathfrak{s} \in \text{Spin}^c(Y)$  (c.f. Theorem 10.1 of [27]), and which admits a natural  $\mathbb{F}[U]$ -equivariant map

$$\pi: HF^\infty(Y, \mathfrak{s}) \longrightarrow HF^+(Y, \mathfrak{s})$$

which is zero in all sufficiently negative degrees and an isomorphism in all sufficiently positive degrees. Note that the quotient

$$HF_{\text{red}}^+(Y) = HF^+(Y) / \pi(HF^\infty(Y))$$

is a finite-dimensional  $\mathbb{F}$ -vector space.

The image of  $\pi$  determines a function

$$d: \text{Spin}^c(Y) \longrightarrow \mathbb{Q}$$

(the “correction terms” of [22]) which associates to each  $\text{Spin}^c$  structure the minimal  $\mathbb{Q}$ -grading of any (non-zero) homogeneous element in  $HF^+(Y, \mathfrak{s})$  in the image of  $\pi$ . Note that orientations play a vital role in Heegaard Floer homology. For example, the correction terms for  $Y$  and its opposite  $-Y$  are related by the formula

$$(5) \quad d(-Y, \mathfrak{s}) = -d(Y, \mathfrak{s}),$$

under a natural identification  $\text{Spin}^c(Y) \cong \text{Spin}^c(-Y)$ .



There is a conjugation symmetry on the space of  $\text{Spin}^c$  structures  $\mathfrak{s} \mapsto \bar{\mathfrak{s}}$ . Heegaard Floer homology is invariant under this symmetry, in the sense that we have a commutative square

$$\begin{array}{ccc} HF^\infty(Y, \mathfrak{s}) & \xrightarrow{\cong} & HF^\infty(Y, \bar{\mathfrak{s}}) \\ \pi \downarrow & & \pi \downarrow \\ HF^+(Y, \mathfrak{s}) & \xrightarrow{\cong} & HF^+(Y, \bar{\mathfrak{s}}) \end{array}$$

In particular, we have that

$$(6) \quad d(Y, \mathfrak{s}) = d(Y, \bar{\mathfrak{s}}).$$

**2.2.  $\mathbb{Z}/2\mathbb{Z}$  gradings.** It is sometimes convenient to consider  $HF^+(Y, \mathfrak{t})$  as a  $\mathbb{Z}/2\mathbb{Z}$ -graded (rather than  $\mathbb{Q}$ -graded) theory. Let  $X$  be a smooth, oriented four-manifold with boundary diffeomorphic to  $Y$ , equipped with a  $\text{Spin}^c$  structure  $\mathfrak{s}$  with  $\mathfrak{s}|_Y = \mathfrak{t}$ . The quantity

$$\frac{c_1(\mathfrak{s})^2 - \sigma(X)}{4},$$

thought of as an element in  $\mathbb{Q}/2\mathbb{Z}$  is easily seen to be independent of the extending four-manifold  $X$ . If  $\xi \in HF_i^+(Y)$  is a non-zero element for some  $i \in \mathbb{Q}$ , then the difference

$$i - \frac{c_1(\mathfrak{s})^2 - \sigma(X)}{4}$$

is an integer (for any choice of  $X$ ), and indeed its parity (which is independent of  $X$ ) is the  $\mathbb{Z}/2\mathbb{Z}$  grading of  $\xi$ .

**2.3. Naturality under cobordisms.** Let  $X$  be a smooth, connected, oriented four-manifold with boundary given by  $\partial X = -Y_0 \cup Y_1$  where  $Y_0$  and  $Y_1$  are connected, oriented three-manifolds. We call such a four-manifold a cobordism from  $Y_0$  to  $Y_1$ . If  $X$  is a cobordism from  $Y_0$  to  $Y_1$ , and  $\mathfrak{s} \in \text{Spin}^c(X)$  is a  $\text{Spin}^c$  structure, then there are naturally induced maps on Heegaard Floer homology which fit into the following diagram:

$$\begin{array}{ccc} HF^\infty(Y_0, \mathfrak{s}_0) & \xrightarrow{F_{X, \mathfrak{s}}^\infty} & HF^\infty(Y_1, \mathfrak{s}_1) \\ \pi_0 \downarrow & & \downarrow \pi_1 \\ HF^+(Y_0, \mathfrak{s}_0) & \xrightarrow{F_{X, \mathfrak{s}}^+} & HF^+(Y_1, \mathfrak{s}_1), \end{array}$$

where here  $\mathfrak{s}_i$  denotes the restriction of  $\mathfrak{s}$  to  $Y_i$ . For fixed  $X$  and  $\xi \in HF^+(Y_0)$ , we have that  $HF_{X, \mathfrak{s}}^+(\xi) = 0$  for all but finitely many  $\mathfrak{s} \in \text{Spin}^c(X)$ , c.f. Theorem 3.3 of [19], and hence there is a well-defined map

$$F_X^+ : HF^+(Y_0) \longrightarrow HF^+(Y_1)$$

defined by

$$F_X^+ = \sum_{\mathfrak{s} \in \text{Spin}^c(X)} F_{X,\mathfrak{s}}^+$$

(note that the same construction does not work for  $HF^\infty$ : for a given  $\xi \in HF^\infty(Y_0)$ , there might be infinitely many different  $\mathfrak{s} \in \text{Spin}^c(X)$  for which  $F_{X,\mathfrak{s}}^\infty(\xi)$  is non-zero).

The map  $\pi: HF^\infty(Y, \mathfrak{s}) \longrightarrow HF^+(Y, \mathfrak{s})$  preserves the  $\mathbb{Q}$ -grading, and moreover, maps induced by cobordisms  $F_{X,\mathfrak{s}}^\circ = F_{X,\mathfrak{s}}^\infty$  or  $F_{X,\mathfrak{s}}^+$  respect the  $\mathbb{Q}$ -grading in the following sense. If  $Y_0$  and  $Y_1$  are rational homology three-spheres, and  $X$  is a cobordism from  $Y_0$  to  $Y_1$ , with  $\text{Spin}^c$  structure  $\mathfrak{s}$ , the map induced by the cobordism maps

$$F_{X,\mathfrak{s}}^\circ: HF_d^\circ(Y_0, \mathfrak{s}_0) \longrightarrow HF_{d+\Delta}^\circ(Y_1, \mathfrak{s}_1),$$

for

$$(7) \quad \Delta = \frac{c_1(\mathfrak{s})^2 - 2\chi(X) - 3\sigma(X)}{4},$$

where here  $HF^\circ = HF^\infty$  or  $HF^+$ ,  $\chi(X)$  denotes the Euler characteristic of  $X$ , and  $\sigma(X)$  denotes its signature. In fact (c.f. Theorem 7.1 of [19]) the  $\mathbb{Q}$  grading is uniquely characterized by the above property, together with the fact that  $d(S^3) = 0$ .

Naturality of the maps induced by cobordisms can be phrased as follows. Suppose that  $W_0$  is a smooth cobordism from  $Y_0$  to  $Y_1$  and  $W_1$  is a cobordism from  $Y_1$  to  $Y_2$ , and suppose moreover that  $Y_i$  are rational homology three-spheres, then

$$F_{W_1}^+ \circ F_{W_0}^+ = F_{W_0 \cup_{Y_1} W_1}^+$$

(c.f. Theorem 3.4 of [19]).

**2.4. Intersection form bounds.** The correction terms of a rational homology three-sphere  $Y$  constrain the intersection forms of smooth four-manifolds which bound  $Y$ , according to the following result, which is analogous to a gauge-theoretic result of Frøyshov [5]:

**Theorem 2.1.** *Let  $Y$  be a rational homology and  $W$  be a smooth four-manifold which bounds  $Y$  with negative-definite intersection form. Then, for each  $\text{Spin}^c$  structure  $\mathfrak{s}$  over  $W$ , we have that*

$$(8) \quad c_1(\mathfrak{s})^2 + b_2(W) \equiv 4d(Y, \mathfrak{s}) \pmod{2},$$

and we have the inequality

$$(9) \quad c_1(\mathfrak{s})^2 + b_2(W) \leq 4d(Y, \mathfrak{s}|_Y).$$

The proof of the above theorem can be found in Theorem 9.6 of [22]. In the case where  $b_1(W) = 0$ , Equation (8) follows easily from the characterization of the  $\mathbb{Q}$  grading (c.f. Equation (7) above), while Inequality (9) follows quickly from the fact (c.f. the proof of Theorem 9.1 in [22]) that if  $X$  is a cobordism with  $b_2^+(X) = b_1(X) = 0$  between rational homology three-spheres  $Y_0$  to  $Y_1$  and  $\mathfrak{s} \in \text{Spin}^c(X)$  then the induced map  $F_{X,\mathfrak{s}}^\infty$

is an isomorphism. In the case where  $b_1(W) > 0$ , one can perform surgery to reduce to the previous case.

By contrast, if  $X$  is a cobordism from  $Y_0$  to  $Y_1$  with  $b_2^+ \neq 0$ , then the induced map on  $HF^\infty$  is trivial, c.f. Lemma 8.2 of [19].

**2.5. Long exact sequences.** Heegaard Floer homology satisfies a surgery long exact sequence, which we state presently. Suppose that  $M$  is a three-manifold with torus boundary, and fix three simple, closed curves  $\gamma_0$ ,  $\gamma_1$ , and  $\gamma_2$  in  $\partial M$  with

$$(10) \quad \#(\gamma_0 \cap \gamma_1) = \#(\gamma_1 \cap \gamma_2) = \#(\gamma_2 \cap \gamma_0) = -1$$

(where here the algebraic intersection number is calculated in  $\partial M$ , oriented as the boundary of  $M$ ), so that  $Y_0$  resp.  $Y_1$  resp.  $Y_2$  are obtained from  $M$  by attaching a solid torus along the boundary with meridian  $\gamma_0$  resp.  $\gamma_1$  resp.  $\gamma_2$ . Note that there are two-handle cobordisms  $W_i$  connecting  $Y_i$  to  $Y_{i+1}$  (where we view  $i \in \mathbb{Z}/3\mathbb{Z}$ ).

**Theorem 2.2.** *Let  $Y_0$ ,  $Y_1$ , and  $Y_2$  be related as above. Then, we have a long exact sequence*

$$\dots \longrightarrow HF^+(Y_0) \longrightarrow HF^+(Y_1) \longrightarrow HF^+(Y_2) \longrightarrow \dots$$

where the maps are induced by from the natural two-handle cobordisms.

The above theorem is proved in Theorem 9.12 of [27].

**2.6. The case where  $b_1(Y) > 0$ .** Although we have restricted our attention mainly to the case of rational homology three-spheres, the construction of Heegaard Floer homology (and in particular Theorem 2.2) works for arbitrary closed, oriented three-manifolds. The rational grading which we discussed here, however, works only for  $\text{Spin}^c$  structures whose first Chern class is torsion. Also, the structure of  $HF^\infty(Y, \mathfrak{s})$  is slightly more intricate. We state now the case where  $H_1(Y; \mathbb{Z}) \cong \mathbb{Z}$ . In this case, the map  $\pi$  is trivial for all  $\text{Spin}^c$  structures with non-zero first Chern class, and in the case where  $c_1(\mathfrak{s}) = 0$ , we have that

$$(11) \quad HF^\infty(Y, \mathfrak{s}) \cong \mathbb{F}[U, U^{-1}] \oplus \mathbb{F}[U, U^{-1}];$$

the first summand has  $\mathbb{Q}$ -grading in  $\frac{1}{2} + 2\mathbb{Z}$ , while the second has  $\mathbb{Q}$ -grading in  $-\frac{1}{2} + 2\mathbb{Z}$ .

**2.7.  $L$ -spaces.** There is a class of three-manifolds for which the Floer homology is particularly simple.

**Definition 2.3.** *A rational homology three-sphere is called an  $L$ -space if for each  $\mathfrak{s} \in \text{Spin}^c(Y)$ , the map  $\pi: HF^\infty(Y) \longrightarrow HF^+(Y)$  is surjective.*

Clearly, for an  $L$ -space, the correction terms determine the Heegaard Floer homology  $HF^+(Y)$ . The reader can find another equivalent definition in [20], which uses a different variant of Heegaard Floer homology. Finally, it should be pointed out that Floer homology depends on the choice of coefficient system, so it would be more precise to call

a three-manifold an  $L$ -space with coefficients in  $\mathbb{F}$  (our underlying coefficient system), but we do not do this here.

Note that if  $Y$  is an  $L$ -space, then so is  $-Y$ .

The following principle gives a plentiful supply of  $L$ -spaces.

**Proposition 2.4.** *Suppose that  $Y_0$ ,  $Y_1$ , and  $Y_2$  are three rational homology three-spheres related by an exact sequence as in Theorem 2.2, and suppose that the number of elements in  $H_1(Y_1; \mathbb{Z})$  is greater than the number of elements in  $H_1(Y_0; \mathbb{Z})$  and  $H_1(Y_2; \mathbb{Z})$ . Suppose moreover that  $Y_0$  and  $Y_2$  are  $L$ -spaces, then so is  $Y_1$ .*

The above proposition is straightforward application of Theorem 2.2, together with the structure of  $HF^\infty$  for a rational homology three-sphere (Equation (4)). Details can be found in Proposition 2.1 of [20].

**2.8. Sharp four-manifolds.** We will describe here a mechanism which is sometimes useful in the calculation of correction terms for  $L$ -spaces. The methods employed here are closely related to discussions in [24] and also Section 3 of [25].

**Definition 2.5.** *Let  $Y$  be an  $L$ -space. A smooth four-manifold  $X$  which bounds  $-Y$  is called sharp if  $b_1(X) = 0$ , the intersection form on  $H^2(X, Y; \mathbb{Z})$  is negative-definite, and if for each  $\mathfrak{t} \in \text{Spin}^c(Y)$ , there is some extension  $\mathfrak{s}$  over  $X$  with the property that*

$$(12) \quad c_1(\mathfrak{s})^2 + b_2(X) = -4d(Y, \mathfrak{t}).$$

Clearly, if  $X$  is sharp, then so is its blowup  $X \# \overline{\mathbb{CP}}^2$ .

**Proposition 2.6.** *Suppose that  $Y_0$ ,  $Y_1$ , and  $Y_2$  are three rational homology three-spheres related by an exact sequence as in Theorem 2.2, and that  $Y_0$  and  $Y_2$  are  $L$ -spaces. Suppose that there is a four-manifold  $X_2$  which is sharp for  $Y_2$ ; and also that  $X_0 = W_0 \cup W_1 \cup X_2$  is sharp for  $Y_0$ . Then  $X_1 = W_1 \cup X_2$  is sharp for  $Y_1$ . (Here,  $W_i$  are the two-handle cobordisms connecting  $Y_i$  to  $Y_{i+1}$ .)*

Suppose that  $X$  is negative-definite four-manifold with  $b_1(X) = 0$  which bounds an  $L$ -space  $-Y$ . Then  $X - B^4$  gives a cobordism from  $Y$  to  $S^3$ . Define  $\mathbb{H}^+(Y; X)$  to be the set of maps

$$\phi: \text{Spin}^c(X) \longrightarrow HF^+(S^3)$$

satisfying the relation that

$$(13) \quad U^{\frac{\langle c_1(\mathfrak{s}), v \rangle + v \cdot v}{2}} \cdot \phi(\mathfrak{s} + \text{PD}(v)) = \phi(\mathfrak{s}),$$

for any  $\mathfrak{s} \in \text{Spin}^c(X)$  and  $v \in H_2(X; \mathbb{Z})$ , for which  $\langle c_1(\mathfrak{s}), v \rangle + v \cdot v \geq 0$ . The  $\mathbb{F}[U]$ -module structure on  $HF^+(S^3)$  gives this set the structure of a  $\mathbb{F}[U]$  module.

There is a map

$$T_X^+: HF^+(Y) \longrightarrow \mathbb{H}^+(Y; X)$$

given by

$$\langle T_X^+(\xi), [\mathfrak{s}] \rangle = F_{X-B^4, \mathfrak{s}}^+(\xi),$$

thinking of  $X - B^4$  as a cobordism from  $Y$  to  $S^3$ . We claim that the image of  $T_X^+$  is in  $\mathbb{H}^+(Y; X) \subset \text{Hom}(\text{Spin}^c(X), HF^+(S^3))$ . This follows at once from the fact that a negative-definite cobordism with  $b_1 = 0$  between a rational homology spheres induces an isomorphism on  $HF^\infty$  for each  $\text{Spin}^c$  structures, whose degree is given by Equation (7).

Given  $d \in \mathbb{Q}$ , let  $\mathbb{H}_d^+(Y; X) \subset \mathbb{H}^+(Y; X)$  denote the set of maps  $\phi$  the property that for all  $\mathfrak{s} \in \text{Spin}^c(X)$  for which  $\phi(\mathfrak{s}) \neq 0$ , we have that  $\phi(\mathfrak{s}) \in HF_i^+(S^3) \subset HF^+(S^3)$  where

$$i - \left( \frac{c_1(\mathfrak{s})^2 + 2 \cdot \chi(X) + 3 \cdot \sigma(X)}{4} \right) = d.$$

Clearly, the restriction of  $T_X^+$  to  $HF_d^+(Y)$  is contained in  $\mathbb{H}_d^+(Y; X)$ .

Clearly,  $X$  is a sharp four-manifold if and only if the map  $T_X^+$  is an isomorphism.

**Lemma 2.7.** *Let  $Y$  be a rational homology three-sphere, and suppose that  $X$  is a smooth four-manifold with negative-definite intersection form. Then, there is an isomorphism*

$$\mathbb{H}^+(Y; X) \cong (\mathbb{F}[U, U^{-1}]/\mathbb{F}[U])^n,$$

where here  $n$  is the number of elements in the image of  $H^2(X; \mathbb{Z})$  inside  $H^2(Y; \mathbb{Z})$ , and the isomorphism is to be viewed as an isomorphism between ungraded  $\mathbb{F}[U]$ -modules.

**Proof.** The isomorphism is induced as follows. Choose for each  $\mathfrak{s} \in \text{Spin}^c(Y)$  which extends over  $X$  an extension  $\tilde{\mathfrak{s}} \in \text{Spin}^c(X)$  for which  $c_1(\tilde{\mathfrak{s}})^2$  is maximal. This can be done since  $X$  has negative-definite intersection form. The map from  $\mathbb{H}^+(Y; X)$  maps  $\phi$  to  $(\phi(\tilde{\mathfrak{s}}_1), \dots, \phi(\tilde{\mathfrak{s}}_n))$ , where  $\{\mathfrak{s}_i\}_{i=1}^n$  is the set of  $\text{Spin}^c$  structures over  $Y$  which extend over  $X$ . It is straightforward to write down an inverse for this map.  $\square$

Let  $Y_0$  and  $Y_1$  be  $L$ -spaces. Let  $W_0$  is a negative-definite cobordism with  $b_1(W_0) = 0$  from  $Y_0$  to  $Y_1$ , and let  $X_1$  be a negative-definite four-manifold with  $b_1(X_1) = 0$  which bounds  $-Y_1$ . Then, there is an induced map:

$$\mathbb{F}_{W_0}^+ : \mathbb{H}^+(Y_0; W_0 \cup X_1) \longrightarrow \mathbb{H}^+(Y_1; X_1).$$

defined by

$$\langle \mathbb{F}_{W_0}^+(\phi), \mathfrak{s}_1 \rangle = \sum_{\{\mathfrak{s} \in \text{Spin}^c(W_0 \cup_{Y_1} X_1) \mid \mathfrak{s}|_{X_1} = \mathfrak{s}_1\}} \phi(\mathfrak{s}).$$

This is easily seen to be a well-defined since  $b_2^+(W_0 \cup X_1) = 0$  and  $\phi$  satisfies Equation (13). The map is natural under cobordisms, in the sense that the following square commutes

$$\begin{array}{ccc} HF^+(Y_0) & \xrightarrow{F_{W_0}^+} & HF^+(Y_1) \\ T_{W_0 \cup_{Y_1} X_1}^+ \downarrow & & \downarrow T_{X_1}^+ \\ \mathbb{H}^+(Y_0; W_0 \cup_{Y_1} X_1) & \xrightarrow{\mathbb{F}_{W_0}^+} & \mathbb{H}^+(Y_1; X_1). \end{array}$$

Commutativity of this square follows at once from the composition law for the maps induced by cobordisms.

**Lemma 2.8.** *Suppose that  $Y_0$ ,  $Y_1$ , and  $Y_2$  are three rational homology three-spheres related by an exact sequence as in Theorem 2.2, and suppose that  $X_2$  is a four-manifold which bounds  $-Y_2$  with  $b_1(X_2) = 0$ , so that  $W_0 \cup W_1 \cup X_2$  is a negative-definite four-manifold. Then, the map*

$$\mathbb{F}_{W_0}^+ : \mathbb{H}^+(Y_0; W_0 \cup W_1 \cup X_2) \longrightarrow \mathbb{H}^+(Y_1; W_1 \cup X_2)$$

*is injective.*

**Proof.** Fix a non-zero  $\phi_0 \in \mathbb{H}^+(Y_0; W_0 \cup W_1 \cup X_2)$ . Its non-triviality means that there is an  $\mathfrak{s} \in \text{Spin}^c(W_0 \cup W_1 \cup X_2)$  with  $\phi_0(\mathfrak{s}) \neq 0$ . Indeed, by multiplying  $\phi_0$  by powers of  $U$  if necessary, we obtain a new element  $\phi \in \mathbb{H}^+(Y_0; W_0 \cup W_1 \cup X_2)$  with the property that  $U \cdot \phi \equiv 0$ , but there is some  $\mathfrak{s} \in \text{Spin}^c(W_0 \cup W_1 \cup X_2)$  with  $\phi(\mathfrak{s}) \neq 0$ . (This follows at once from Lemma 2.7.)

The kernel of the map  $H^2(W_0 \cup W_1 \cup X_2; \mathbb{Z}) \longrightarrow H^2(W_1 \cup X_2; \mathbb{Z})$  is generated by  $\text{PD}(\Sigma_0)$ , where  $\Sigma_0 \in H_2(W_0, Y_0; \mathbb{Z}) \cong \mathbb{Z}$  is a generator. Moreover, the kernel of the map  $H^2(W_0 \cup W_1 \cup X_2; \mathbb{Z}) \longrightarrow H^2(X_2; \mathbb{Z})$  is generated by two homology classes  $\text{PD}(\Sigma_0)$  and  $\text{PD}(e)$ , and  $e \in H_2(W_0 \cup W_1; \mathbb{Z})$  with  $\Sigma_0 \cdot e = 1$ , and  $e \cdot e = -1$ . (Indeed, the class  $e$  can be represented by an embedded two-sphere.)

Fix  $\mathfrak{s}_0$  so that  $\phi(\mathfrak{s}_0) \neq 0$ . Indeed, by subtracting off  $\text{PD}(e)$  if necessary, we can assume without loss of generality that  $\langle \phi(\mathfrak{s}_0), e \rangle = -1$ . Choose a maximal integer  $a$  so that  $\phi(\mathfrak{s}_0 + a \cdot \text{PD}(\Sigma_0) + a \cdot \text{PD}(e)) \neq 0$ . This exists since  $\phi$  has finite support (this in turn follows from Equation (13), together with the fact that  $W_0 \cup W_1 \cup X_2$  has negative-definite intersection form). Note that  $\langle c_1(\mathfrak{s}_0 + (a+1) \cdot \text{PD}(\Sigma_0) + a \cdot \text{PD}(e)), e \rangle = +1$ , and hence it follows (from Equation (13), together with the choice of  $a$ ) that

$$(14) \quad \phi(\mathfrak{s}_0 + (a+1) \cdot \text{PD}(\Sigma_0) + a \cdot \text{PD}(e)) = \phi(\mathfrak{s}_0 + (a+1) \cdot \text{PD}(\Sigma_0) + (a+1) \cdot \text{PD}(e)) = 0.$$

Thus, for  $\mathfrak{s} = \mathfrak{s}_0 + a \cdot \text{PD}(\Sigma_0) + a \cdot \text{PD}(e)$ , we have that

$$(15) \quad \langle c_1(\mathfrak{s}), e \rangle = -1, \quad \phi(\mathfrak{s}) \neq 0, \quad \phi(\mathfrak{s} + \text{PD}(\Sigma_0)) = 0.$$

Clearly, in the orbit  $\mathfrak{s} + \mathbb{Z} \cdot \text{PD}(\Sigma_0)$ , only  $c_1(\mathfrak{s})$  and  $c_1(\mathfrak{s} + \text{PD}(\Sigma_0))$  have evaluation  $\pm 1$  on  $e$ , and hence only  $\mathfrak{s}$  and  $\mathfrak{s} + \text{PD}(\Sigma_0)$  have the possibility of having non-trivial value under  $\phi$ . However, Equation (15) proves that

$$\langle \mathbb{F}_{W_0}^+(\phi), \mathfrak{s}|_{W_1 \cup X_2} \rangle = \sum_{b \in \mathbb{Z}} \phi(\mathfrak{s} + b \cdot \text{PD}(\Sigma_0)) = \phi(\mathfrak{s}) + \phi(\mathfrak{s} + \text{PD}(\Sigma_0)) \neq 0.$$

□

**Proof of Proposition 2.6.** Consider the diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & HF^+(Y_0) & \xrightarrow{F_{W_0}} & HF^+(Y_1) & \xrightarrow{F_{W_1}} & HF^+(Y_2) \longrightarrow 0 \\
 & & \cong \downarrow T_{X_0}^+ & & \downarrow T_{X_1}^+ & & \cong \downarrow T_{X_2}^+ \\
 0 & \longrightarrow & \mathbb{H}^+(Y_0; X_0) & \xrightarrow{\mathbb{F}_{W_0}^+} & \mathbb{H}^+(Y_1; X_1) & \xrightarrow{\mathbb{F}_{W_1}^+} & \mathbb{H}^+(Y_2; X_2),
 \end{array}$$

where here the top row is exact. According to Lemma 2.8,  $\mathbb{F}_{W_0}^+$  is an injection. A straightforward diagram-chase now establishes that  $T_{X_1}^+$  is injective.

By Lemma 2.7,  $T_{X_1}^+$  must be an isomorphism. This follows from the following observation: suppose

$$f: \left( \frac{\mathbb{F}[U, U^{-1}]}{\mathbb{F}[U]} \right)^a \longrightarrow \left( \frac{\mathbb{F}[U, U^{-1}]}{\mathbb{F}[U]} \right)^b$$

is an injective map of  $\mathbb{F}[U]$  modules for some  $a \geq b$ , then  $a = b$  and indeed  $f$  is an isomorphism. This can be seen by restricting to the kernel of  $U^n$  (for all  $n$ ), and appealing to the corresponding fact for finite-dimensional vector spaces.  $\square$

## 3. FIRST APPLICATIONS OF FLOER HOMOLOGY

In the introduction, there was no reference to  $\text{Spin}^c$  structures. To see why these can be eliminated, recall that if  $H^2(Y; \mathbb{Z})$  has odd order then the map which sends  $\mathfrak{s} \in \text{Spin}^c(Y)$  to half its first Chern class  $c_1(\mathfrak{s})/2$  induces an isomorphism

$$\text{Spin}^c(Y) \longrightarrow H^2(Y; \mathbb{Z}).$$

If  $K$  is a knot in  $S^3$ , then  $H^2(\Sigma(K); \mathbb{Z})$  has odd order, so we can use the above map to identify  $\text{Spin}^c$  structures with integral two-dimensional cohomology classes. Note that under the above identification, the conjugation symmetry on  $\text{Spin}^c(Y)$  is identified with multiplication by  $-1$ .

With this said, we get the following rather quick consequence of Lemma 1.5 and Theorem 2.1, according to which the correction terms for  $\Sigma(K)$  give an obstruction for  $K$  to have unknotting number one.

**Theorem 3.1.** *If  $K$  is a knot with unknotting number one, then there is some isomorphism  $\phi: \mathbb{Z}/D\mathbb{Z} \xrightarrow{\cong} H^2(\Sigma(K); \mathbb{Z})$  with the property that at least one of the two conditions holds:*

- for all  $i \in \mathbb{Z}/D\mathbb{Z}$ ,

$$\gamma_D(i) \equiv d(\Sigma(K), \phi(i)) \pmod{2} \quad \text{and} \quad \gamma_D(i) \leq d(\Sigma(K), \phi(i)),$$

- for all  $i \in \mathbb{Z}/D\mathbb{Z}$ ,

$$\gamma_D(i) \equiv -d(\Sigma(K), \phi(i)) \pmod{2} \quad \text{and} \quad \gamma_D(i) \leq -d(\Sigma(K), \phi(i)).$$

**Proof.** If  $K$  has unknotting number equal to one, then by Montesinos' trick (Lemma 1.5),  $\Sigma(K) \cong \pm S^3_{D/2}(C)$  for some knot  $C \subset S^3$ . Thus,  $\pm \Sigma(K)$  bounds a four-manifold  $W$  with intersection form

$$\begin{pmatrix} -n & 1 \\ 1 & -2 \end{pmatrix},$$

where here  $D = 2n - 1$ . The theorem now is a direct application of Theorem 2.1 in this context.  $\square$

Of course, to apply Theorem 3.1 meaningfully, one must calculate the correction terms for  $\Sigma(K)$ . When  $K$  is an alternating knot, the correction terms can be calculated in terms of the Goeritz matrix for  $K$ . Furthermore, when  $K$  is an alternating knot,  $HF^+(\Sigma(K))$  has a particularly simple form.

To this end, we take the Goeritz form, where the knot is colored according to the coloring conventions as specified in Figure 1.

**Proposition 3.2.** *If  $K$  is an alternating knot, then  $\Sigma(K)$  is an  $L$ -space; and indeed, if  $Q$  denotes the Goeritz form of  $K$  and  $q$  the induced map as described earlier, then there*



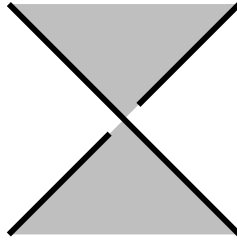


FIGURE 1. Coloring conventions for alternating knots.

is an affine identification

$$\phi: \text{Coker}(q) \xrightarrow{\cong} \text{Spin}^c(\Sigma(K))$$

taking zero to the unique spin structure, with the property that

$$M_Q(\xi) = d(\Sigma(K), \phi(\xi)).$$

The proof of the above result can be found in [25] (combining Proposition 3.3 and Theorem 3.4 both in [25]), but we sketch the argument here for the reader's convenience.

Starting from a connected, alternating projection of a link  $L$ , one constructs a four-manifold  $X_L$  bounding  $\Sigma(L)$  as follows. For all but one vertex in the black graph, we associate a one-handle to attach to the four-ball. In terms of Kirby calculus, to each of these vertices, we associate a dotted unknot. Then, to each crossing (edge in the black graph), we associate an unknot with framing  $-1$  which links the two dotted unknots which it connects. The intersection form of  $X_L$  is the Goeritz form for  $L$ , as can be seen after performing handleslides of the  $-1$ -framed unknots around the circuits given by vertices in the white graph. (This description can be easily seen to agree with the one given in [25], where the Goeritz form is described with respect to a different basis.)

Fix a crossing for some regular alternating projection of  $L$ , and let  $L_0$  and  $L_1$  be the two links formed by resolving the crossing in two ways as pictured in Figure 2, and suppose that both  $L_0$  and  $L_1$  have connected projection (if such a point cannot be found, then  $L$  is a projection of the unknot). It is easy to see that  $-\Sigma(L_0)$ ,  $-\Sigma(L_1)$ , and  $-\Sigma(L)$  are related as in Theorem 2.2 (with  $Y_0 = -\Sigma(L_0)$ ,  $Y_1 = -\Sigma(L)$ , and  $Y_2 = -\Sigma(L_1)$ ). Now, by induction on the number of crossings, together with Proposition 2.4, shows that  $\Sigma(L)$  is an  $L$ -space. Indeed, observing that  $W_0 \cup W_1 \cup X_{L_1} = X_{L_0} \# \overline{\mathbb{CP}}^2$ , the same inductive argument (now using Proposition 2.6) can be used to show that  $X_L$  is sharp for  $\Sigma(L)$ .

Note that Theorem 3.1 and Proposition 3.2 suffice to establish Conditions (1) and (2) in Theorem 1.1. However, to establish Condition (3), we need an analogous symmetry for the correction terms of  $L$ -spaces which are obtained as  $(2n-1)/2$ -surgery on a knot in  $S^3$ .

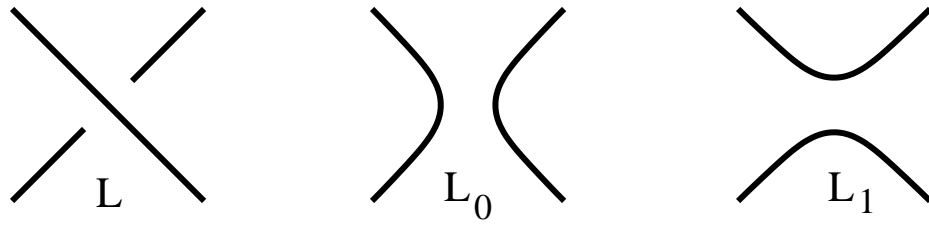


FIGURE 2. **Skein moves.** Let  $L$  be an alternating link, and  $L_0$  and  $L_1$  be the links obtained by resolving some fixed crossing of  $L$  according to the illustrated conventions.

4.  $L$ -SPACE SURGERIES

We now turn our attention to the following symmetry result which quickly provides the missing piece of the proof of Theorem 1.1.

Note that there is a natural cobordism  $W$  from  $S^3$  to  $S^3_{-(2n-1)/2}(C)$ , whose Kirby calculus picture is given by  $C$  with framing  $-n$ , and a linking unknot with framing  $-2$ . This cobordism can naturally be broken into a pair of cobordisms  $W_1$  from  $S^3$  to  $S^3_{-n}(C)$ , followed by a cobordism  $W_2$  from  $S^3_{-n}(C)$  to  $S^3_{-(2n-1)/2}(C)$ . The intersection form for  $W_1 \cup_{S^3_{-n}(C)} W_2$  is given by  $R_{-(2n-1)/2}$ . Such an identification is specified by choosing a generator  $F$  of  $H_2(W_1; \mathbb{Z}) \cong \mathbb{Z}$ , or, equivalently, an orientation on  $C$ . We fix this additional datum for the purposes of the rest of the present section (though this choice does not affect the final results, in view of the conjugation symmetry of the correction terms).

For any knot  $C \subset S^3$ , we have an isomorphism

$$\phi: \mathbb{Z}/(2n-1)\mathbb{Z} \longrightarrow H^2(S^3_{-(2n-1)/2}(C))$$

which takes  $i$  to  $i \cdot \text{PD}[F]|_{S^3_{-(2n-1)/2}(C)}$ . (We suppress the knot  $C$  from the notation, but bear in mind that sometimes we use this map  $\phi$  for  $C$  and at other times we use  $\phi$ .) In the following statement, we use  $c_1(\mathfrak{s})/2$  to identify  $\text{Spin}^c(S^3_{-(2n-1)/2}(C)) \cong H^2(S^3_{-(2n-1)/2}(C); \mathbb{Z})$ .

**Theorem 4.1.** *Let  $O$  be the unknot, and let  $C \subset S^3$  be a knot with the property that for some  $n > 1$ ,  $S^3_{-(2n-1)/2}(C)$  is an  $L$ -space, and suppose moreover that*

$$d(S^3_{-(2n-1)/2}(C), 0) = d(S^3_{-(2n-1)/2}(O), 0).$$

*Write  $n = 2k$  or  $2k + 1$  for integral  $k$ . Then, under the isomorphism  $\phi$ , we have that*

$$\begin{aligned} (16) \quad & d(S^3_{-(2n-1)/2}(C), \phi(i)) - d(S^3_{-(2n-1)/2}(O), \phi(i)) \\ &= d(S^3_{-(2n-1)/2}(C), \phi(2k-i)) - d(S^3_{-(2n-1)/2}(O), \phi(2k-i)) \end{aligned}$$

*for  $i = 1, \dots, k$ ; and when  $n = 2k + 1$ , Equation (16) also holds for  $i = 0$ .*

We give the proof at the end of the present section. Indeed, Theorem 4.1 follows quickly from another result (Theorem 4.2) which identifies the difference in correction terms for the  $-(2n-1)/2$ -surgery with a corresponding difference for the  $-n$ -surgery. To state this, we introduce some notation.

Let  $\{F, S\}$  be the basis for  $H_2(W; \mathbb{Z})$  corresponding to the knot  $C$  and the linking unknot (here  $S$  is represented by a sphere with square  $-2$  and  $S \cdot F = +1$ ). Let  $(a, b)$  be the cohomology class whose evaluation on  $F$  and  $S$  are  $a$  and  $b$  respectively.

We now give an ordered list of  $2n - 1$  characteristic vectors for the intersection form of  $W$ , in two cases depending on the parity of  $n$ . If  $n$  is even, write  $n = 2k$  and let

$$(17) \quad \kappa_i = \begin{cases} (2i, 0) & \text{for } -k \leq i \leq k \\ (-4k + 2i, 2) & \text{for } k < i \leq 2k - 1 \\ (2i - 4k + 2, -2) & \text{for } 2k \leq i \leq 3k - 2 \end{cases}$$

while if  $n$  is odd, write  $n = 2k + 1$ , and let

$$(18) \quad \kappa_i = \begin{cases} (1 + 2i, -2) & \text{for } 0 \leq i \leq k \\ (2i - 4k - 1, 0) & \text{for } k + 1 \leq i \leq 3k + 1 \\ (2i - 8k - 3, 2) & \text{for } 3k + 2 \leq i \leq 4k \end{cases}$$

For  $i = 0, \dots, 2n - 2$ , let  $w_i \in \text{Spin}^c(S^3_{-(2n-1)/2}(C))$  be the  $\text{Spin}^c$  structure which can be extended to  $\mathfrak{s}_i \in \text{Spin}^c(W)$  with  $c_1(\mathfrak{s}_i) = \kappa_i$ . Let  $v_i \in \text{Spin}^c(S^3_{-n}(C))$  denote the restriction of  $\mathfrak{s}_i$  to  $S^3_{-n}(C)$ .

Of course, the  $\{v_i\}_{i=0}^{2n-2}$  are not all distinct  $\text{Spin}^c$  structures on  $S^3_{-n}(C)$ : the  $\text{Spin}^c$  structure is determined by the first coordinate of  $\kappa_i \pmod{2n}$ .

**Theorem 4.2.** *Assume that  $C \subset S^3$  is a knot with the property that for some  $n > 1$   $S^3_{-(2n-1)/2}(C)$  is an L-space. Then so is  $S^3_{-n}(C)$ . Moreover, if*

$$d(S^3_{-(2n-1)/2}(C), w_0) = d(S^3_{-(2n-1)/2}(O), w_0),$$

then we have for all  $i = 0, \dots, 2n - 2$ ,

$$d(S^3_{-n}(C), v_i) - d(S^3_{-n}(O), v_i) = d(S^3_{-(2n-1)/2}(C), w_i) - d(S^3_{-(2n-1)/2}(O), w_i).$$

The proof will be given in the end of the section.

**Lemma 4.3.** *Let  $C$  be any knot in  $S^3$ . For  $i \in \mathbb{Z}/(2n - 1)\mathbb{Z}$ ,*

$$(19) \quad 0 \leq d(S^3_{-n}(C), v_i) - d(S^3_{-n}(O), v_i) \leq d(S^3_{-(2n-1)/2}(C), w_i) - d(S^3_{-(2n-1)/2}(O), w_i).$$

**Proof.** The argument which establishes Theorem 2.1 actually proves the following result: suppose that  $W$  is a cobordism from  $Y_0$  to  $Y_1$ , both of which are rational homology three-spheres, and suppose that the intersection form of  $W$  is negative-definite, then for each  $\mathfrak{s} \in \text{Spin}^c(W)$ ,

$$(20) \quad d(Y_0, \mathfrak{s}|_{Y_0}) + \left( \frac{c_1^2(\mathfrak{s}) - 2\chi(W) - 3\sigma(W)}{4} \right) \leq d(Y_1, \mathfrak{s}|_{Y_1}).$$

(This can alternatively thought of as a consequence of Theorem 2.1, together with the additivity of the correction terms under connected sums, c.f. Theorem 4.3 of [22].)

This establishes the existence of constants  $k(n, i)$  with the property that for any knot  $C$ ,

$$d(S^3_{-n}(C), w_i) + k(n, i) \leq d(S^3_{-(2n-1)/2}(C), v_i).$$

Indeed, we can choose the constants so that equality holds when  $C = O$ . This follows readily from the fact that the negative-definite plumbing for  $S^3_{-(2n-1)/2}(O)$  is sharp, which in turn is an easy consequence of Proposition 2.6, see also [24]. This establishes the second inequality. The first is proved along the same lines.  $\square$

The following result about  $L$ -spaces will be useful to us, as well. A more general result is established in Section 7 of [12] in the context of the Seiberg-Witten monopole Floer homology (though the proof adapts readily to the case at hand).

**Proposition 4.4.** *Suppose that  $S^3_{-(2n-1)/2}(C)$  is an  $L$ -space. Then so is  $S^3_{-n}(C)$ .*

**Lemma 4.5.** *If  $C$  is any knot in  $S^3$ , then the map induced by the cobordism*

$$HF^+(S^3_{(2n-1)/2}(C)) \longrightarrow HF^+(S^3_n(C))$$

*is surjective.*

**Proof.** First, we argue that for all  $n \geq 2$ , the natural map  $HF^+(S^3_{n-1}(C)) \longrightarrow HF^+(S^3_n(C))$  is injective for all  $n$ . This follows at once from the surgery exact sequence

$$\dots \xrightarrow{D} HF^+(S^3_{n-1}(C)) \xrightarrow{A} HF^+(S^3_n(C)) \longrightarrow HF^+(S^3) \xrightarrow{D} \dots,$$

together with the observation that  $D$  is induced by a cobordism  $W$  with  $b_2^+(W) = 1$ , and hence the induced map from  $HF^+(S^3)$  is trivial. Thus, we see that the surgery long exact sequence becomes a short exact sequence.

Next, consider the long exact sequence

$$\xrightarrow{B} HF^+(S^3_{n-1}(C)) \longrightarrow HF^+(S^3_{(2n-1)/2}(C)) \longrightarrow HF^+(S^3_n(C)) \xrightarrow{B}$$

Observe that the cobordism  $A \circ B$  admits an alternative factorization as the standard (two-handle) cobordism from  $S^3_n(C)$  to  $S^3_n(C) \# (S^2 \times S^1)$ , followed by another cobordism (in which the generator of  $S^2 \times S^1$  becomes null-homologous). From this, it follows readily that the induced  $A \circ B = 0$  (c.f. Lemma 2.9 of [29]). Since  $A$  is injective, it follows that  $B = 0$ . Exactness now shows that the stated map is surjective.  $\square$

**Proof of Proposition 4.4.** Lemma 4.5 proves at once that if  $S^3_{2n-1/2}(C_1)$  is an  $L$ -space, then so is  $S^3_n(C_1)$ . Note that  $Y$  is an  $L$ -space iff  $-Y$  is, and also if  $C_1$  is the mirror of  $C$ , then  $S^3_r(C_1) = -S^3_{-r}(C)$  for any rational number  $r$ . Thus, the claim follows.  $\square$

If  $Y$  is a rational homology three-sphere, write

$$d(Y) = \sum_{\mathfrak{s} \in \text{Spin}^c(Y)} d(Y, \mathfrak{s}).$$

**Proposition 4.6.** *Let  $C \subset S^3$  be a knot, and suppose that  $S^3_{-(2n-1)/2}$  is an  $L$ -space. Then,*

$$d(S^3_{-(2n-1)/2}(C)) - 2 \cdot d(S^3_{-n}(C)) = d(S^3_{-(2n-1)/2}(O)) - 2 \cdot d(S^3_{-n}(O)).$$

The above follows from a more general result (Proposition 4.7) proved in Subsection 4.1.

With the preliminaries in place, we can now turn to the proofs of the two theorems stated in the beginning of the present section.

**Proof of Theorem 4.2.** Since we assume that  $S^3_{-(2n-1)/2}(C)$  is an  $L$ -space, Proposition 4.4 guarantees that  $S^3_{-n}(C)$  is, too. Our goal is to show that if  $S^3_{-(2n-1)/2}(C)$  is an  $L$ -space, then the inequalities from Lemma 4.3 are all equalities:

$$(21) \quad d(S^3_{-(2n-1)/2}(C), v_i) - d(S^3_{-(2n-1)/2}(O), v_i) = d(S^3_{-n}(C), w_i) - d(S^3_{-n}(O), w_i).$$

for  $i = 0, \dots, 2n - 2$ .

By inspecting Equations (17) and (18), observe that the list  $\{v_i\}_{i=0}^{2n-2}$  contains each  $\text{Spin}^c$  structure over  $S^3_{-n}(C)$  twice, except for one. Specifically, in the case where  $n = 2k$ , the  $\text{Spin}^c$  structure which appears only once in the list is  $v_0$ . Combining Lemma 4.3 with the hypothesis that  $d(S^3_{-(2n-1)/2}(C), w_0) = d(S^3_{-(2n-1)/2}(O), w_0)$ , we conclude that

$$(22) \quad 0 = d(S^3_{-n}(C), v_0) - d(S^3_{-n}(O), v_0) \quad \text{when } n = 2k.$$

Similarly, when  $n = 2k + 1$ , the  $\text{Spin}^c$  structure appearing only once on the list is  $v_{2k}$ . It is easy to see that  $v_{2k} = \overline{v_0}$ , and hence combining the conjugation symmetry (Equation (6)) with Lemma 4.3, we get that

$$(23) \quad 0 = d(S^3_{-n}(C), v_{2k}) - d(S^3_{-n}(O), v_{2k}) \quad \text{when } n = 2k + 1.$$

Note that Lemma 4.3 gives  $2n - 1$  inequalities which we must prove are all equalities. Adding up these inequalities, we get

$$(24) \quad \begin{aligned} & \sum_{i=0}^{2n-2} (d(S^3_{-(2n-1)/2}(C), w_i) - d(S^3_{-(2n-1)/2}(O), w_i)) \\ & \geq \sum_{i=0}^{2n-2} (d(S^3_{-n}(C), v_i) - d(S^3_{-n}(O), v_i)) \\ & = 2 \sum_{\mathfrak{t} \in \text{Spin}^c(S^3_{-n}(K))} (d(S^3_{-n}(C), \mathfrak{t}) - d(S^3_{-n}(O), \mathfrak{t})), \end{aligned}$$

where here the last equation uses the fact that every  $\mathfrak{t} \in \text{Spin}^c(S^3_{-n}(K))$  is represented twice amongst the  $\{v_i\}_{i=0}^{2n-2}$ , except for one  $v_j = v_0$  or  $v_{2k}$ , depending on the parity of  $n$ ) for which we already know that  $d(S^3_{-n}(C), v_j) - d(S^3_{-n}(O), v_j) = 0$  (Equations (22) and (23)).

Proposition 4.6 now forces the inequality to be equality; and this in turn implies that each of the  $2n - 1$  individual inequalities in Inequality (19) are also equalities.  $\square$

**Proof of Theorem 4.1.** By inspecting Equations (17) and (18), we see that for  $i = 0, \dots, 2n - 2$  we have that  $c_1(w_i)/2 = \phi(i)$ . Moreover, another glance at those definitions reveals that when  $n = 2k$ ,  $v_i$  and  $v_{2k-i}$  are conjugate  $\text{Spin}^c$  structures for  $i = 1, \dots, k - 1$ . Similarly, when  $n = 2k + 1$ , then  $v_i$  and  $v_{2k-i}$  are conjugate for  $i = 0, \dots, k - 1$ . It follows that

$$d(S_{-n}^3(C), v_i) - d(S_{-n}^3(O), v_i) = d(S_{-n}^3(C), v_{2k-i}) - d(S_{-n}^3(O), v_{2k-i}).$$

Thus, the theorem follows from Theorem 4.2.  $\square$

**4.1. Euler characteristics.** We now return to the proof of Proposition 4.6 stated above. Indeed, we prove a more general statement. To give the statement, we introduce some notation. If  $Y$  is a rational homology three-sphere and  $k$  is some constant, then we define

$$HF_{\leq k}^+(Y) = \bigoplus_{\{d \in \mathbb{Q} \mid d \leq k\}} \bigoplus_{\mathfrak{s} \in \text{Spin}^c(Y)} HF_d^+(Y, \mathfrak{s}).$$

In the case where  $H_1(Y; \mathbb{Z}) \cong \mathbb{Z}$ , we let

$$HF_{\leq k}^+(Y) = \bigoplus_{\{\mathfrak{s} \in \text{Spin}^c(Y) \mid c_1(\mathfrak{s}) \neq 0\}} HF^+(Y, \mathfrak{s}) \oplus \bigoplus_{\{d \in \mathbb{Q} \mid d \leq k\}} HF_d^+(Y, \mathfrak{s}_0),$$

where  $c_1(\mathfrak{s}_0) = 0$ .

In this latter case, the structure of  $HF^\infty$  (c.f. Equation (11)) ensures that for all sufficiently large integers  $N$ ,  $\chi(HF_{\leq 2N + \frac{1}{2}}^+(Y))$  is independent of  $N$ . We denote this constant by  $\chi^{\text{trunc}}(HF^+(Y))$ .

**Proposition 4.7.** *For fixed relatively prime integers  $p$  and  $q$  with  $p > 0$ , there is a constant  $k(p, q)$  with the property that for any knot  $C \subset S^3$ ,*

$$\sum_{i \in \mathbb{Z}/p\mathbb{Z}} \left( \chi(HF_{\text{red}}^+(S_{p/q}^3(C), i)) - \frac{d(S_{p/q}^3(C), i)}{2} \right) - q \cdot \chi^{\text{trunc}}(HF^+(S_0^3(C))) = k(p, q)$$

We break the proof into two steps.

**Lemma 4.8.** *Let  $p$  and  $q$  be relatively prime integers with  $p \geq 0$ . There is a constant  $k_1(p, q)$  with the property that for any knot  $C \subset S^3$  and any sufficiently large integer  $N$ ,*

$$\begin{aligned} (25) \quad & \chi(HF_{\leq 2N}^+(S_{p/q}^3(C))) - N \cdot p \\ &= \sum_{\mathfrak{s} \in \text{Spin}^c(S_{p/q}^3(C))} \left( \chi(HF_{\text{red}}^+(S_{p/q}^3(C), \mathfrak{s})) - \frac{d(S_{p/q}^3(C), \mathfrak{s})}{2} \right) + k_1(p, q). \end{aligned}$$

**Proof.** Let  $Y = S^3_{p/q}(C)$ . For sufficiently large  $N$ ,  $HF^+_{\text{red}}(Y)$  is contained in  $HF^+_{\leq 2N}(Y)$ . Over  $\mathbb{F}$ , we have a splitting

$$HF^+_{\leq 2N}(Y) \cong HF^+_{\text{red}}(Y) \oplus (\text{Im}\pi \cap HF^+_{\leq 2N}(Y)).$$

But it follows readily from the structure of  $HF^\infty(Y)$  (c.f. Equation (4)) that

$$\begin{aligned} \chi(\text{Im}\pi \cap HF^+_{\leq 2N}(Y)) &= \sum_{\mathfrak{s} \in \text{Spin}^c(Y)} \# \{[d(Y, \mathfrak{s}), 2N] \cap (d + 2\mathbb{Z}) \subset \mathbb{Q}\} \\ &= \sum_{\{\mathfrak{s} \in \text{Spin}^c(Y)\}} \left( N + 1 - \lceil \frac{d(Y, \mathfrak{s})}{2} \rceil \right), \end{aligned}$$

where here  $\lceil x \rceil$  denotes the smallest integer greater than or equal to  $x$ . Thus, since the number of  $\text{Spin}^c$  structures on  $Y$  is  $p$ , we get that

$$\begin{aligned} \chi(HF^+_{\leq 2N}(Y)) - N \cdot p &= \sum_{\mathfrak{s} \in \text{Spin}^c(Y)} \left( HF^+_{\text{red}}(Y, \mathfrak{s}) - \lceil \frac{d(Y, \mathfrak{s})}{2} \rceil + 1 \right) \\ &= \sum_{\mathfrak{s} \in \text{Spin}^c(Y)} \left( HF^+_{\text{red}}(Y, \mathfrak{s}) - \frac{d(Y, \mathfrak{s})}{2} \right) + \sum_{\mathfrak{s} \in \text{Spin}^c(Y)} \left( \frac{d(Y, \mathfrak{s})}{2} - \lceil \frac{d(Y, \mathfrak{s})}{2} \rceil + 1 \right). \end{aligned}$$

but this last sum is easily seen to depend on  $d(Y, \mathfrak{s})$  only modulo  $2\mathbb{Z}$ ; i.e. it depends only on the linking form of  $Y = S^3_{p/q}(C)$ , which in turn is independent of the knot  $C$  (depending only on  $p$  and  $q$ ).  $\square$

**Lemma 4.9.** *Given a pair of non-negative, relatively prime integers  $p$  and  $q$ , there is a constant  $k_2(p, q)$  with the property that for any knot  $C \subset S^3$ , there is a natural number  $N_0$  with the property that for all  $N \geq N_0$ ,*

$$\chi(HF^+_{\leq 2N}(S^3_{p/q}(C))) - N \cdot p = q \cdot \chi^{\text{trunc}}(HF^+(S^3_0(C))) + k_2(p, q).$$

**Proof.** We will use induction on  $p + q$ , together with the fact that for any pair  $(p, q)$  of relatively prime, non-negative integers with  $p + q > 1$ , there are two pairs of non-negative, relatively integers  $(p_0, q_0)$  and  $(p_2, q_2)$  with the properties that

$$(26) \quad p_0 \cdot q - p \cdot q_0 = -1$$

$$(27) \quad (p, q) = (p_0, q_0) + (p_2, q_2)$$

In the basic case of the lemma where  $p + q = 1$ , we consider cases where  $(p, q) = (1, 0)$  or  $(0, 1)$ . In both cases, the lemma is clear.

For arbitrary relatively prime, non-negative integers  $(p, q)$ , find non-negative integers  $(p_0, q_0)$  and  $(p_2, q_2)$  satisfying Equations (26) and (27). Let  $Y_0 = S^3_{p_0/q_0}(C)$ ,  $Y_1 =$



$S_{p/q}^3(C)$ , and  $Y_2 = S_{p_2/q_2}^3(C)$ . Those equations guarantee that we have a long exact sequence

$$\dots \longrightarrow HF^+(Y_0) \xrightarrow{f_0} HF^+(Y_1) \xrightarrow{f_2} HF^+(Y_2) \xrightarrow{f_3} \dots,$$

and also that the lemma is known for  $(p_0, q_0)$  and  $(p_2, q_2)$ . We assume first that that  $p_0 \neq 0$ .

When  $N$  is sufficiently large, the restriction  $g_0$  of  $f_0$  to  $HF_{\leq 2N}^+(Y_0)$  is contained in  $HF_{\leq 2N+\frac{1}{4}}^+(Y_1)$ , the restriction of  $f_2$  to  $HF_{\leq 2N+\frac{1}{4}}^+(Y_1)$  is contained in  $HF_{\leq 2N+\frac{1}{2}}^+(Y_2)$ , and the finally, the restriction of  $f_3$  to  $HF_{\leq 2N+\frac{1}{2}}^+(Y_2)$  is contained in  $HF_{\leq 2N}^+(Y_0)$ . This follows at once from the grading shift formula, Equation (7): we have that  $\chi(W_i) = 1$  and  $\sigma(W_i) = -1$  for  $i = 0, 1$ ; while the cobordism  $W_2$  induces the trivial map on  $HF^\infty$  since  $b_2^+(W_0) = 1$ .

Choosing  $N$  as above, consider the diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \xrightarrow{g_2} & HF_{\leq 2N}^+(Y_0) & \xrightarrow{g_0} & HF_{\leq 2N+\frac{1}{4}}^+(Y_1) & \xrightarrow{g_1} & HF_{\leq 2N+\frac{1}{2}}^+(Y_2) \xrightarrow{g_2} \dots \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \xrightarrow{f_2} & HF^+(Y_0) & \xrightarrow{f_0} & HF^+(Y_1) & \xrightarrow{f_1} & HF^+(Y_2) \xrightarrow{f_2} \dots \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \xrightarrow{h_2} & HF_{> 2N}^+(Y_0) & \xrightarrow{h_0} & HF_{> 2N+\frac{1}{4}}^+(Y_1) & \xrightarrow{h_1} & HF_{> 2N+\frac{1}{2}}^+(Y_2) \xrightarrow{h_2} \dots \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0, \end{array}$$

where the columns are exact. Note that the rows are not necessarily exact (except for the middle one), however all three rows can be thought of as chain complexes. We denote these three rows by  $\mathcal{R}_1$ ,  $\mathcal{R}_2$ , and  $\mathcal{R}_3$ . Since  $\mathcal{R}_2$  is exact, it follows that  $H_*(\mathcal{R}_1) \cong H_*(\mathcal{R}_3)$ .

We claim that  $H_*(\mathcal{R}_3)$  is independent of  $C$  and  $N$  (provided the latter is sufficiently large). To see this, observe that we have a diagram

$$\begin{array}{ccccccc} \dots & \xrightarrow{h_2^\infty} & HF_{> 2N}^\infty(Y_0) & \xrightarrow{h_0^\infty} & HF_{> 2N+\frac{1}{4}}^\infty(Y_1) & \xrightarrow{h_1^\infty} & HF_{> 2N+\frac{1}{2}}^\infty(Y_2) \xrightarrow{h_0^\infty} \dots \\ & & \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\ \dots & \xrightarrow{h_2} & HF_{> 2N}^+(Y_0) & \xrightarrow{h_0} & HF_{> 2N+\frac{1}{4}}^+(Y_1) & \xrightarrow{h_1} & HF_{> 2N+\frac{1}{2}}^+(Y_2) \xrightarrow{h_0} \dots, \end{array}$$

where here  $h_0$  is the sum over all  $\mathfrak{s} \in \text{Spin}^c(W_0)$  of the projections of the induced maps on  $HF^\infty$ ; e.g. letting

$$\Pi_{>2N+\frac{1}{2}}: HF^\infty(Y_1) \longrightarrow HF_{>2N+\frac{1}{2}}^\infty(Y_1)$$

denote the projection, we let  $h_0^\infty$  be the restriction to  $HF_{>2N}(Y_0)$  of

$$\sum_{\mathfrak{s} \in \text{Spin}^c(W_0)} \Pi_{>2N+\frac{1}{2}} \circ F_{W_0, \mathfrak{s}}^\infty.$$

The map  $h_i^\infty$  are defined similarly. Note that  $h_2^\infty = 0$ , since the map induced by  $W_2$  has  $b_2^+(W_2) = 1$  (in general, this is proved in (c.f. Lemma 8.2 of [19], though in the present case it follows quickly from the dimension formula: the cobordism  $W_2$  reverses the  $\mathbb{Z}/2\mathbb{Z}$  grading, but  $HF^\infty$  is supported entirely in even gradings).

So far we have established that for all sufficiently large  $N$ ,

$$\chi(H_*(\mathcal{R}_1)) = \chi(HF_{\leq 2N}^+(Y_0)) - \chi(HF_{\leq 2N+\frac{1}{4}}^+(Y_1)) + \chi(HF_{\leq 2N+\frac{1}{2}}^+(Y_2))$$

is independent of  $C$  and  $N$  (provided that the latter is sufficiently large); but it is also clear that for sufficiently large  $N$ ,

$$\chi(HF_{\leq 2N+\frac{1}{4}}^+(Y_1)) = \chi(HF_{\leq 2N}^+(Y_1)) + c_3 \quad \text{and} \quad \chi(HF_{\leq 2N+\frac{1}{2}}^+(Y_2)) = \chi(HF_{\leq 2N}^+(Y_2)) + c_4,$$

with constants  $c_3$  and  $c_4$  again depending only on  $(p_0, q_0)$  and  $(p_2, q_2)$  respectively. Combining all the constants, we establish the inductive step, at least in the case where  $p_0$  is non-zero.

In the case where  $p_0 = 0$ , the above argument works with slight modification. In this case, we cyclically order the three three-manifolds  $(Y_0, Y_1, Y_2)$  so that  $Y_1$  has  $b_1(Y_1) = 1$ . In this case, the dimension shifts works slightly differently:  $\sigma(W_0) = \sigma(W_1) = 0$  and hence, we compare  $HF_{\leq 2N}^+(Y_0)$ ,  $HF_{\leq 2N+\frac{1}{2}}^+(Y_1)$ , and  $HF_{\leq 2N+1}^+(Y_2)$  (to see that the maps induced by  $W_2$  carry  $HF_{\leq 2N+1}^+(Y_2)$  for sufficiently large  $N$ , note that Equation (7) ensures that the the shift in grading is at least zero, and also that  $HF_{\leq 2N+1}^+(Y_2) = HF_{\leq 2N}^+(Y_2)$  for sufficiently large  $N$ :  $HF^\infty$  is supported only in even degrees). With these minor modifications, the previous argument establishes the inductive step in the remaining case, as well.  $\square$

**Proof of Proposition 4.7.** When  $p$  and  $q$  are non-negative, this is a combination of Lemmas 4.8 and 4.9. This could be proved either by running the induction from Lemma 4.9 to show that Lemma 4.9 still holds in the case where  $p > 0$  and  $q \leq 0$ ; or alternatively, deducing this from the previous case, using the facts that  $S_{p/q}^3(C) = -S_{-p/q}^3(r(C))$  (where  $r$  denotes reflection),  $\chi(HF_{\text{red}}^+(-Y)) = -\chi(HF_{\text{red}}^+(Y))$ , and the fact that  $\chi^{\text{trunc}}(HF^+(S_0^3(C))) = \chi^{\text{trunc}}(HF^+(S_0^3(r(C))))$ , all of which are established in [27].  $\square$

**Proof of Proposition 4.6.** Note that if  $S^3_{-(2n-1)/2}(C)$  is an  $L$ -space, then according to Proposition 4.4, so is  $S^3_{-(2n-1)/2}(C)$ , and in particular  $\chi(HF^+_{\text{red}}(S^3_{-(2n-1)/2}(C), i)) = 0$  and  $\chi(HF^+_{\text{red}}(S^3_{-n}(C), i)) = 0$ . The rest is a direct application of Proposition 4.7.  $\square$

## 5. PROOF OF THEOREM 1.1.

**Proof of Theorem 1.1.** If  $K$  has unknotting number one, by Montesinos' trick (Lemma 1.5), we know that  $\Sigma(K) = \pm S_{D/2}^3(C)$ . After reflecting  $K$  if necessary, we can achieve that  $\Sigma(K) = S_{-D/2}^3(C)$ . According to Proposition 3.2,  $\Sigma(K)$  is an  $L$ -space, and we have an isomorphism

$$\phi: \text{Coker}(q) \xrightarrow{\cong} H^2(\Sigma(K); \mathbb{Z})$$

with the property that

$$M_Q(\xi) = d(\Sigma(K), \phi(\xi)).$$

(Note that  $\Sigma(r(K)) = -\Sigma(K)$ , and hence the reflection has the effect of reversing the signs of the correction terms, c.f. Equation (5); this is responsible for the sign  $\epsilon$  appearing in the statement of the theorem.)

Now the expression of  $S_{-D/2}^3 \cong \Sigma(K)$  gives us an identification  $\mathbb{Z}/D\mathbb{Z} \cong \text{Coker}(q)$ . Thus, according to Theorem 2.1, we get both

$$\begin{aligned} \gamma_D(i) &\equiv M_Q(\phi(i)) \pmod{2\mathbb{Z}} \\ \gamma_D(i) &\leq M_Q(\phi(i)), \end{aligned}$$

where as usual  $\gamma_D(i) = M_{R_D}(i)$ .

We claim that  $M_{R_D}(i) = d(S_{-D/2}^3(O), i)$ . This can be seen, for example, by taking a two-bridge knot whose branched double-cover is  $S_{-D/2}^3(0)$ , and applying Proposition 3.2.

Note that for  $D = 2n - 1$ ,

$$M_{R_D}(0) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{1}{2} & \text{if } n \text{ is even.} \end{cases}$$

Thus, when  $|M_{Q_K}(0)| \leq \frac{1}{2}$ , it follows that  $M_{R_D}(0) = M_{Q_K}(0)$ , verifying the hypothesis of Theorem 4.1. The symmetry  $T_\phi(i) = T_\phi(2k - i)$  is now a direct consequence of Theorem 4.1.  $\square$

We note that the obstruction given by Theorem 1.1 does not depend on the choice of the alternating projection of  $K$ , since, according to Proposition 3.2,  $M_Q$  is a topological invariant of the oriented branched double-cover. Note also that if we reflect  $K$  (or, equivalently, use the black instead of the white graphs), this has the effect of reversing the orientation of  $\Sigma(K)$ , and hence the corresponding correction terms (as given by  $M_Q$  for the Goeritz form, c.f. Proposition 3.2) all get multiplied by  $-1$ . This is compensated by our freedom in choosing  $\epsilon = \pm 1$ . We return to this point in Section 8.

## 6. CALCULATIONS FOR ALTERNATING KNOTS

We explain now how to apply Theorem 1.1 in detail.

Given an alternating knot of determinant  $D$ , we start by writing down its  $m \times m$  Goeritz matrix  $G$ . (In practice, it can be useful to reflect the knot if necessary to minimize the number of white regions.) Suppose that  $H_1(\Sigma(K); \mathbb{Z})$  is cyclic or, equivalently, that  $\text{Coker}(q) \cong \mathbb{Z}/D\mathbb{Z}$ .

Next, we find the function  $M_Q: \text{Coker}(q) \rightarrow \mathbb{Q}$ . Two vectors  $v_1, v_2 \in \mathbb{Z}^m$  correspond to equivalent vectors in  $\text{Coker}(q)$  when  $G^{-1}(v_1 - v_2) \in \mathbb{Z}^m$ . The induced quadratic form  $Q^*$  in this basis is represented by  $(v, w) \mapsto v^t \cdot G^{-1} \cdot w$ . We claim that characteristic vectors in  $V^* \cong \mathbb{Z}^m$  which achieve maximal length are contained in the finite set

$$\{v = (v_1, \dots, v_m) \in \mathbb{Z}^m \mid \text{for } i = 1, \dots, m, |v_i| \leq |G_{i,i}| \text{ and } v_i \equiv G_{i,i} \pmod{2}\};$$

if a vector lies outside that set, it is straightforward to find another equivalent vector  $v'$  with larger length.

Thus, by performing a finite set of calculations, we end up with a list of  $D$  vectors in  $\mathbb{Z}^m$  which maximize their length in their equivalence class. We order these vectors  $\{x(i)\}_{i=0}^{D-1}$ , so that  $x(i)$  represents  $i \in \mathbb{Z}/D\mathbb{Z}$  under some isomorphism of  $\text{Coker}(q) \cong \mathbb{Z}/D\mathbb{Z}$ . Next, form the vector  $\{a_i\}_{i=0}^{D-1}$ , where

$$4a_i = x(i)^t \cdot G^{-1} \cdot x(i) + m.$$

Note that this ordering of the vectors in  $A$  is not canonical: it is canonical only up to reordering given by multiplication by the units in  $\mathbb{Z}/D\mathbb{Z}$ . According to Proposition 3.2, the vector  $A$  contains the correction terms for  $\Sigma(K)$ .

Next, we calculate  $B_i = \gamma_D(i)$ . This is straightforward: let  $4B_i = \kappa(i)^t \cdot R_D^{-1} \cdot \kappa(i) + 2$ , where the  $\kappa(i)$  are given in order in Equations (17) and (18).

Now for each automorphism  $\phi$  of  $\mathbb{Z}/D\mathbb{Z}$  and sign  $\epsilon = \pm 1$ , we form the vector  $C_i = -B_i - \epsilon A_{\phi(i)}$ . We call such a vector  $C_i$  a *matching* for the knot  $K$ . The set of matchings for an alternating knot  $K$  is a knot invariant. We call a matching *even* if it consists of even integers. We call a matching *positive* if it consists of non-negative rational numbers. Finally, writing  $D = 4k \pm 1$ , we call a matching *symmetric* if it satisfies the symmetry  $C_i = C_{2k-i}$  for  $1 < i < k$  and also for  $i = 0$  when  $D = 4k + 1$ . Note that matchings always satisfy the symmetry  $C_i = C_{D-i}$ , and hence the matching is determined by  $\{C_i\}_{i=0}^{n-1}$ , where  $D = 2n - 1$ .

In this language, Theorem 1.1 says that if  $K$  is an alternating knot with unknotting number equal to one, then there is at least one matching  $C$  which is positive and even. Moreover, if  $|A_0| \leq \frac{1}{2}$ , then there is also an even, positive, and symmetric matching. (Note that the condition that  $|A_0| \leq \frac{1}{2}$  is equivalent to the existence of some matching for which  $C_0 = 0$ .) Note also that all the knots we consider in this section satisfy the condition that  $|A_0| \leq \frac{1}{2}$ .

We now turn to the applications of Theorem 1.1 to knots with small crossing numbers.

**Proof of Corollary 1.2.**

We start with the knot  $8_{10}$  pictured in Figure 3. Its Goeritz form is represented by the matrix

$$G = \begin{pmatrix} -4, & 1, & 1 \\ 1, & -2, & 1 \\ 1, & 1, & -5 \end{pmatrix},$$

whose determinant is  $-27$ .

Following the above procedure, we find the maximal squares of lengths of the vectors in  $\mathbb{Z}^3$  in each equivalence class or, more precisely, divide these numbers by four, add  $3/4$ , and then order according to the group structure of  $\mathbb{Z}/27\mathbb{Z}$ . This gives us the ordered list of numbers:

$$A = \begin{pmatrix} -\frac{1}{2} & \frac{25}{54} & -\frac{35}{54} & \frac{1}{6} & -\frac{59}{54} & -\frac{23}{54} & \frac{1}{6} & \frac{37}{54} & -\frac{47}{54} \\ -\frac{1}{2} & -\frac{11}{54} & \frac{1}{54} & \frac{1}{6} & \frac{13}{54} & \frac{13}{54} & \frac{1}{6} & \frac{1}{54} & -\frac{11}{54} \\ -\frac{1}{2} & -\frac{47}{54} & \frac{37}{54} & \frac{1}{6} & -\frac{23}{54} & -\frac{59}{54} & \frac{1}{6} & -\frac{35}{54} & \frac{25}{54} \end{pmatrix}.$$

We have written these in an order compatible with the isomorphism of  $\text{Coker}(q) \cong \mathbb{Z}/27\mathbb{Z}$ , where the first term corresponds to the spin structure (and the  $i^{\text{th}}$  term corresponds to the cohomology class  $(0, 0, i)$ ).

We compare this ordered list with the list

$$B = \begin{pmatrix} \frac{1}{2} & \frac{23}{54} & \frac{11}{54} & -\frac{1}{6} & -\frac{37}{54} & -\frac{73}{54} & -\frac{13}{6} & -\frac{169}{54} & -\frac{121}{54} \\ -\frac{3}{2} & -\frac{49}{54} & -\frac{25}{54} & -\frac{1}{6} & -\frac{1}{54} & -\frac{1}{54} & -\frac{1}{6} & -\frac{25}{54} & -\frac{49}{54} \\ -\frac{3}{2} & -\frac{121}{54} & -\frac{169}{54} & -\frac{13}{6} & -\frac{73}{54} & -\frac{37}{54} & -\frac{1}{6} & \frac{11}{54} & \frac{23}{54} \end{pmatrix},$$

which are  $\gamma_{27}(i)$  for  $i = 0, \dots, 26$ .

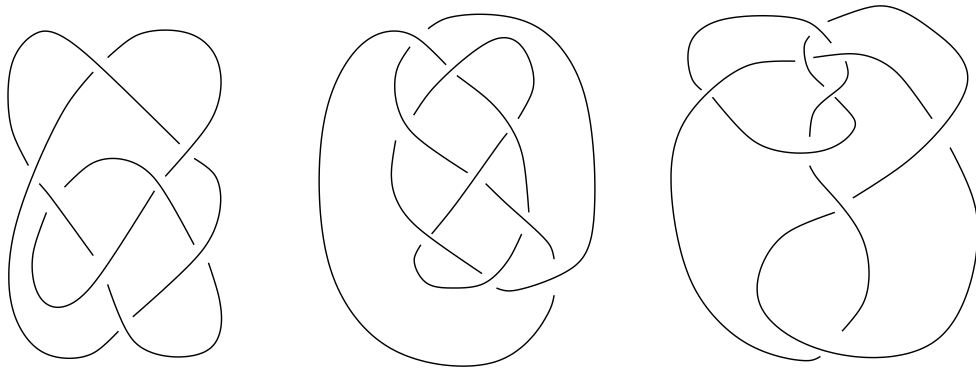


FIGURE 3. The knots  $8_{10}$ ,  $9_{29}$ , and  $9_{32}$ .

By comparing  $A_0$  and  $B_0$  we see that an even matching (for some automorphism  $\phi$  and  $\epsilon = \pm 1$ ) can exist only when  $\epsilon = 1$ . By inspection, we see that there are only two possible automorphisms  $\phi$  of  $\mathbb{Z}/27\mathbb{Z}$  (multiplication by  $\pm 5$ ) for which  $C_i = -B_i - A_{\phi(i)}$  is a non-negative, even integer for  $i = 0, \dots, 26$ . For both of these,  $C_i = -B_i - A_{\phi(i)}$  ( $T_{1,\phi}$ ) is given by the list

$$0, 0, 0, 0, 0, 2, 2, 4, 2, 2, 2, 0, 0, 0, 0, 0, 2, 2, 2, 4, 2, 2, 0, 0, 0, 0.$$

The matching evidently fails the symmetry:  $T_{1,\phi}(4) \neq T_{1,\phi}(10)$ . So there are no even, positive, symmetric matchings. Thus, by Theorem 1.1,  $8_{10}$  cannot have unknotting number equal to one.

We abbreviate this data somewhat. If  $C$  is a matching, we will list only the first  $n$  terms (where  $D = 2n - 1$ ), dropping all the initial and final zero terms, and indicating the  $k^{th}$  term in bold face. For example, we indicate the even, positive matching for  $8_{10}$  by

$$2, 2, \mathbf{4}, 2, 2, 2.$$

The fact that this matching is asymmetric is now obvious from this notation.

For  $9_{29}$ , there is only one possible choice of  $\epsilon$  for which we can find a  $\phi$  satisfying Condition (1); and for that choice, there are four possible  $\phi$  satisfying Conditions (1) and (2), all of which give the same matching:

$$2, 2, 2, 2, 4, 4, 6, \mathbf{6}, 6, 4, 4, 4, 2, 2, 2, 2.$$

For  $9_{32}$ , again there is only one even, positive matching:

$$2, 2, 2, 2, 4, 4, 6, 6, \mathbf{8}, 6, 6, 4, 4, 4, 2, 2, 2, 2.$$

Since both of the above matchings are not symmetric, Theorem 1.1 shows that the knots do not have unknotting number equal to one.

On the other hand, it is clear from their pictures that the three knots considered here can be unknotted in two steps.  $\square$

**6.1. Classification of  $\leq 9$ -crossing knots with  $u = 1$ .** Note that this classification already follows from Corollary 1.2, together with previously known results as explained in the introduction. However, for completeness we include a classification here using Murasugi's bound, the cyclicity of  $H_1(\Sigma(K); \mathbb{Z})$ , and Theorem 1.1.

First observe that the following knots fail Condition (1) in Theorem 1.1, (i.e. they admit no even matchings, in the terminology from the beginning of this section):

$$(28) \quad 7_4, 8_8, 8_{16}, 9_{15}, 9_{17}, 9_{31}.$$

For example, a Goeritz matrix for  $8_{16}$  is given by

$$G = \begin{pmatrix} -4 & 1 & 1 \\ 1 & -4 & 1 \\ 1 & 1 & -3 \end{pmatrix}.$$





condition is now evident. Note that the knots  $10_{58}$  and  $10_{77}$  appear twice on this list, since they have two distinct matchings.

$10_{48}$ :	2 2 2 2 4 4 4 <b>6</b> 6 4 2 2 2 2
$10_{51}$ :	2 2 2 4 4 4 6 6 8 <b>8</b> 6 6 6 4 4 4 2 2 2 2 2
$10_{52}$ :	2 2 2 2 4 4 4 6 6 <b>8</b> 6 6 4 4 4 2 2 2 2 0 2
$10_{54}$ :	2 2 2 4 4 6 <b>6</b> 4 4 4 2 2 2 2 2
$10_{57}$ :	2 2 2 2 2 4 4 6 6 6 8 8 <b>10</b> 8 8 8 6 6 4 4 4 2 2 2 2 2
$10_{58}$ :	2 2 2 4 4 4 6 6 6 <b>8</b> 6 6 4 4 4 2 2 2
$10'_{58}$ :	2 2 2 2 2 4 4 4 6 6 6 <b>8</b> 8 6 6 4 4 4 2 2 2
$10_{64}$ :	2 2 4 4 4 6 <b>6</b> 6 4 4 2 2 2 2 2
$10_{70}$ :	2 2 2 2 4 4 4 6 6 8 <b>8</b> 8 6 6 4 4 4 2 2 2 2 0 2
$10_{77}$ :	2 2 2 4 4 6 6 6 <b>8</b> 6 6 4 4 4 2 2 2 2 2
$10'_{77}$ :	2 2 2 4 4 4 6 6 <b>8</b> 6 6 6 4 4 4 2 2 2 0
$10_{79}$ :	2 0 2 2 2 4 4 4 6 6 <b>8</b> 6 6 4 4 4 2 2 2
$10_{81}$ :	2 2 2 2 4 4 4 4 6 6 6 8 8 10 <b>10</b> 10 8 8 6 6 4 4 4 4 2 2 2 2
$10_{83}$ :	2 2 2 2 2 4 4 4 6 6 8 8 8 <b>10</b> 10 8 8 6 6 6 4 4 4 2 2 2 2 2
$10_{87}$ :	2 2 2 2 2 4 4 4 6 6 6 8 8 10 <b>10</b> 8 8 8 6 6 4 4 4 2 2 2 2 2
$10_{90}$ :	2 0 2 2 2 2 4 4 4 6 6 6 8 8 <b>10</b> 8 8 6 6 6 4 4 4 2 2 2 2
$10_{93}$ :	2 2 2 2 4 4 6 6 6 8 <b>8</b> 8 6 6 4 4 4 4 2 2 2 2 2
$10_{94}$ :	2 2 2 4 4 4 6 6 8 8 <b>8</b> 8 6 6 6 4 4 4 2 2 2 2 2
$10_{96}$ :	2 2 2 2 4 4 4 4 6 6 6 8 8 10 10 <b>12</b> 10 10 8 8 6 6 4 4 4 4 2 2 2 2
$10_{110}$ :	2 2 2 2 4 4 4 4 6 6 8 8 10 <b>10</b> 10 8 8 6 6 6 4 4 4 4 2 2 2 2
$10_{112}$ :	2 2 2 2 2 4 4 4 6 6 8 8 8 10 <b>10</b> 10 10 8 8 6 6 6 4 4 4 2 2 2 2 2
$10_{117}$ :	2 2 2 2 2 4 4 4 4 6 6 8 8 8 10 10 12 <b>12</b> 12 10 10 10 8 8 6 6 6 4 4 4 4 2 2 2 2 2

**6.3. Classification of alternating 10-crossing knots with  $u = 1$ .** Corollary 1.3, together with known results (c.f. [8]) suffice to classify all alternating 10-crossing knots with  $u = 1$ . Indeed, for alternating 10-crossing knots, the previously known cases can be reproved using only the Murasugi bound, cyclicity of  $H_1(\Sigma(K); \mathbb{Z})$ , and Theorem 1.1.

Specifically, consider those alternating, 10-crossing knots with  $|\sigma(K)| \leq 2$  and cyclic  $H_1(\Sigma(K); \mathbb{Z})$ , but which were not covered by Corollary 1.3. Again, various knots fail various of the tests in Theorem 1.1.

The following knots admit no even matching:

$$10_3, 10_{19}, 10_{20}, 10_{24}, 10_{29}, 10_{36}, 10_{40}, 10_{65}, 10_{69}, 10_{89}, 10_{97}, 10_{108}, 10_{122}.$$

The following knots have positive, even matchings, but none of these matchings is symmetric:

$$10_4, 10_{11}, 10_{12}, 10_{13}, 10_{15}, 10_{16}, 10_{22}, 10_{28}, 10_{34}, 10_{35}, 10_{37}, 10_{38}, 10_{41}, 10_{43}, 10_{45}, 10_{115}.$$

## 7. NON-ALTERNATING 10-CROSSING KNOTS AND THE PROOF OF COROLLARY 1.4

The knots described in Corollary 1.4 are not alternating. However, we claim that their branched double-covers are  $L$ -spaces, and hence we can adapt the principles used in the proof of Theorem 1.1 (Montesinos' trick, followed by Theorem 4.1) only using correction terms for the  $\Sigma(K)$ , in place of lengths of vectors of the Goeritz matrix. The key problem remains to verify that  $\Sigma(K)$  are  $L$ -spaces as claimed, and then calculating the correction terms for  $\Sigma(K)$ .

Some of these knots are Montesinos knots, and their branched double-covers are Seifert fibered spaces. Hence the Heegaard Floer homology can be calculated using techniques from [24], as explained in Subsection 7.1. The remaining cases are handled in Subsection 7.2.

**7.1. Corollary 1.4: the Montesinos cases.** The knots in the list

$$(30) \quad 10_{125}, 10_{126}, 10_{130}, 10_{135}, 10_{138}.$$

are *Montesinos knots*, knots whose branched double-covers are Seifert fibered spaces; in fact, the branched double-covers are the spaces with Seifert invariants

$$(31) \quad \left(-2, \frac{1}{2}, \frac{1}{3}, \frac{4}{5}\right), \left(-2, \frac{1}{2}, \frac{2}{3}, \frac{1}{5}\right), \left(-2, \frac{1}{2}, \frac{2}{3}, \frac{3}{7}\right), \left(-2, \frac{1}{2}, \frac{1}{3}, \frac{2}{7}\right), \left(-2, \frac{1}{2}, \frac{2}{5}, \frac{2}{5}\right)$$

respectively. Here our conventions on Seifert invariants are as follows:  $(b, \frac{\beta_1}{\alpha_1}, \dots, \frac{\beta_n}{\alpha_n})$  are the Seifert invariants for the three-manifold obtained as surgery on a configuration consisting of a central circle, with surgery coefficient  $b$ , and a collection of circles linking the central circle, with surgery coefficients  $-\frac{\alpha_i}{\beta_i}$ .

For Seifert a fibered rational homology three-sphere  $Y$ , an algorithm is given in [24] which can be used to determine if  $Y$  is an  $L$ -space. Specifically, we start with a negative-definite plumbing diagram for  $\pm Y$ , let  $V$  denote the lattice generated by the vertices, and let  $Q$  denote the induced bilinear form. As presented,  $V$  has a preferred basis given by the vertices of the plumbing graph. We consider the equivalence relation on characteristic vectors in  $V^*$  generated by

$$\kappa \sim \kappa \pm q(v) \quad \text{if } v \in V \text{ is a preferred basis vector with } \pm \langle \kappa, v \rangle = Q(v, v).$$

It is clear that if  $\kappa \sim \lambda$  then  $Q^*(\kappa \otimes \kappa) = Q^*(\lambda \otimes \lambda)$ . Let  $X$  denote the number of equivalence classes with the property that each  $\kappa$  representing a given element of  $X$  satisfies the bound

$$|\langle \kappa, v \rangle| \leq Q(v, v)$$

for each preferred basis vector  $v \in V$ . Clearly, the number of elements in  $X$  is at least as large as the number of elements in  $\text{Coker } q$ . In fact, according to [24], the number of elements of  $X$  is the rank of  $\text{Ker } U \subset HF^+(-Y)$ , while it is elementary to see that the number of elements in the cokernel of  $q$  is identified with the number of elements in  $H^2(Y; \mathbb{Z})$ . Thus, if the number of elements in  $X$  agrees with the number of elements

in  $H^2(Y; \mathbb{Z})$ , then  $Y$  is an  $L$ -space. Moreover, in [24], it is shown that under these circumstances, the map

$$M_Q: \text{Coker}(q) \longrightarrow \mathbb{Q}$$

agrees with the correction terms for  $-Y$  under a suitable identification of  $\text{Coker}(q)$  with  $\text{Spin}^c(Y)$ .

A straightforward calculation shows that all of the Seifert fibered spaces in List (31) satisfy this criterion, and hence are  $L$ -spaces, and hence Theorem 4.1 can be used to deduce the existence of a symmetry for some matching of the vector of correction terms with our usual vector  $B$ .

For instance, for  $\Sigma(10_{125})$ , a matrix representing  $Q$  in the preferred basis is given by

$$\begin{pmatrix} -2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

Calculating the function  $M_Q$ , and ordering the elements of  $\text{Spin}^c(Y)$  in a manner compatible with the group structure, we get the vector  $A$ :

$$A = \left( \frac{1}{2}, \frac{35}{22}, \frac{19}{22}, \frac{7}{22}, -\frac{1}{22}, -\frac{5}{22}, -\frac{5}{22}, -\frac{1}{22}, \frac{7}{22}, \frac{19}{22}, \frac{35}{22} \right).$$

Comparing against the corresponding vector  $B$ , we find that there are no even, positive, symmetric matchings. Indeed, proceeding in a like manner for all knots in List (30), we have that all even, positive matchings are given by the following table:

$10_{125}$ :	<b>2</b> <b>2</b>
$10_{126}$ :	<b>2</b> <b>2</b>
$10_{130}$ :	<b>2</b> <b>2</b> <b>2</b> <b>2</b>
$10_{135}$ :	<b>2</b> <b>2</b> <b>2</b> <b>4</b> <b>4</b> <b>4</b> <b>2</b> <b>2</b>
$10'_{135}$ :	<b>2</b> <b>2</b> <b>2</b> <b>2</b> <b>4</b> <b>4</b> <b>4</b> <b>2</b> <b>2</b> <b>2</b>
$10_{138}$ :	<b>2</b> <b>2</b> <b>2</b> <b>4</b> <b>4</b> <b>4</b> <b>2</b> <b>2</b> <b>2</b> <b>2</b>

None of the above matchings is symmetric, and hence applying Montesinos' trick together with Theorem 4.1 in the usual manner, we see that none of these knots has unknotting number equal to one.

**7.2. Corollary 1.4: the remaining cases.** The knots  $10_{148}$ ,  $10_{151}$ ,  $10_{158}$ , and  $10_{162}$  are not Montesinos knots. However, their branched double-covers are  $L$ -spaces, and indeed we use a refinement of Proposition 2.6 to calculate the correction terms of  $\Sigma(K)$ . Thus, we must construct sharp four-manifolds which bound the knots in this list.

A knot projection of  $K$  specifies a four-manifold  $Z_K$  which bounds  $\Sigma(K)$ . Starting from the white graph, we draw an unknot with framing 0 for all but one of the vertices

in the white graph. Next, we associate to each edge in the white graph an unknot which links the two 0-framed unknots corresponding to the two vertices. This unknot is given framing  $+1$  if the crossing is consistent with the coloring convention illustrated in Figure 1, and it is given framing  $-1$  if it does not. This framing  $\pm 1$  will be called the sign of the edge. From  $Z_K$  we can obtain another four-manifold  $X_K$  by blowing down all of the unknots with framing  $\pm 1$  corresponding to the white graph. The remaining link is obtained as a plumbing of unknots, with intersection form specified by the Goeritz form for the projection of  $K$ . Specifically, the two-dimensional homology of  $X_K$  is generated by the vertices of the white graph, modulo the relation

$$\sum_{v \in V} v = 0,$$

where here  $V$  denotes the vertices of the white graph. Also,  $Q(v \otimes v)$  is given by minus the sum of the signs of all the edges leaving  $v$ , while if  $v \neq w$ , then  $Q(v \otimes w)$  is the sum of the signs of all the edges connecting  $v$  and  $w$ . (In the alternating case, this construction can be seen to be equivalent to the one given in Section 3.)

For example, for the knot  $K = 10_{148}$  pictured in Figure 4, we obtain a Kirby picture of  $X_K$  which is a plumbing of unknots, as pictured in Figure 5, after we blow down the circle labeled with  $r = -1$ . Ignoring this circle (i.e. performing  $r = \infty$  surgery), we obtain a picture for the branched double-cover of  $\Sigma(K_1)$ , while setting  $r = 0$ , we get a picture for a branched double-cover of  $\Sigma(K_0)$ , where here  $K_0$  and  $K_1$  are the knots obtained by resolving either of the intersection points in the oval marked by  $x$  in Figure 4. Indeed, blowing down the unknot with framing  $+1$ , we obtain a four-manifold with intersection form given by the matrix

$$G = \begin{pmatrix} -4 & 3 & 1 & 0 & 1 \\ 3 & -5 & 0 & 0 & 0 \\ 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 1 & 0 & 0 & 0 & r-1 \end{pmatrix}.$$

In fact, let  $X_2$  denote the four-manifold obtained by ignoring the unknot with framing  $r-1$  (whose intersection form is the top  $4 \times 4$  block of  $G$ ). Now,  $-\Sigma(K_0)$ ,  $-\Sigma(K)$ , and  $-\Sigma(K_1)$  are related by a surgery long exact sequence, with cobordisms

$$W_0: -\Sigma(K_0) \longrightarrow -\Sigma(K) \quad \text{and} \quad W_1: -\Sigma(K) \longrightarrow -\Sigma(K_1).$$

Moreover, if we blow down the obvious sphere of square  $-1$  in the composite cobordism  $X_0 = W_0 \cup W_1 \cup X_2$ , we obtain a four-manifold  $X'_0$  with intersection form given by  $G$  with  $r = 0$ , while  $X_1 = W_1 \cup X_2$  has intersection form given by  $G$  with  $r = -1$ . Our aim is to show that  $X_1$  is sharp.

To this end, we claim that the branched double-cover of  $K_0$  is an  $L$ -space, and indeed that the four-manifold with intersection form given by  $G$  with  $r = -1$  is sharp. Indeed, the branched double-cover of  $\Sigma(K_0)$  is gotten by  $+8$  surgery on the right-handed

trefoil, an  $L$ -space whose correction terms can be calculated for example, using [24]; or alternatively [22]. Comparison with  $M_Q$  for the quadratic form for  $G$  above with  $r = 0$ , we conclude that the four-manifold  $X_0$  is sharp.

Moreover,  $K_1$  is an alternating link and the top  $4 \times 4$  submatrix  $G$  is the Goeritz matrix for an alternating projection. Thus,  $X_2$  is sharp.

Having verified that  $-\Sigma(K_0)$  (a space with 8  $\text{Spin}^c$  structures) and  $-\Sigma(K_1)$  (a space with 23  $\text{Spin}^c$  structures) are  $L$ -spaces, it follows from Proposition 2.4 that  $-\Sigma(K)$  (a space with 31  $\text{Spin}^c$  structures) is an  $L$ -space as well. Indeed, by Proposition 2.6, it also now follows that the matrix  $G$  with  $r = -1$  can be used to calculate the correction terms  $A$  for  $\Sigma(K)$ .

Again, there is a unique even, positive matching, and it is given by

$$10_{148} : 2, 2, 4, 2, 2, 2.$$

We proceed similarly for the other three knots, which are pictured in Figure 7.

For the knot  $K = 10_{151}$ , the Kirby calculus description of  $Z_K$  contains a certain chain unknots as illustrated in Figure 8 with  $r = -1$ , corresponding to the circled region in the diagram for  $10_{151}$ . Blowing down the two  $-1$ -framed and all the  $+1$ -framed ones unknots gives a four-manifold with indefinite intersection form. Instead, we blow down all the  $+1$ -framed unknots, and then perform two handleslides and two handle

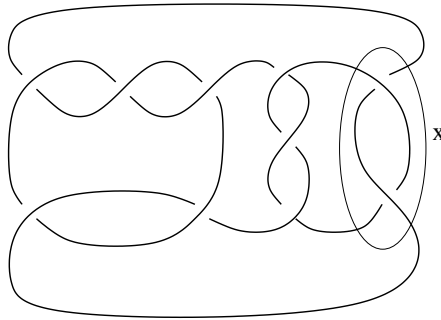


FIGURE 4. The knot  $10_{148}$ .

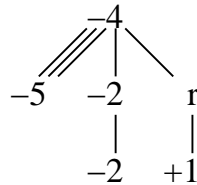


FIGURE 5. Kirby calculus pictured for  $\Sigma(10_{148})$ . The integers here are framings on unknots, and the lines represent linkings between the unknots. Setting  $r = -1$ , we obtain a picture of a four-manifold which bounds  $\Sigma(10_{148})$ .

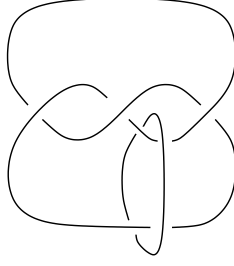


FIGURE 6. **A link with determinant 8.** This is the link obtained by vertically resolving one of the crossings in the oval  $x$  for  $10_{148}$  in Figure 4.

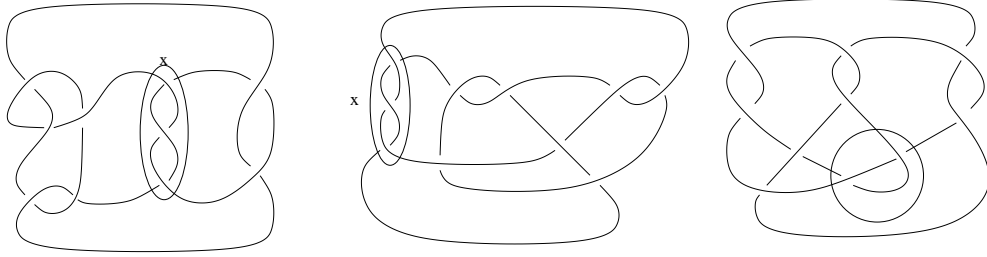


FIGURE 7. **The knots  $10_{151}$ ,  $10_{158}$ , and  $10_{162}$  respectively.**

cancellations, after trading the 0-framed unknots for one-handles. This replaces the chain with a single unknot with framing  $r - 2$ .

In this manner, we obtain a four-manifold  $X(r)$  whose intersection form is given by:

$$G = \begin{pmatrix} -3 & 1 & 1 & 1 & 1 \\ 1 & -2 & 1 & 0 & 0 \\ 1 & 1 & -4 & 0 & 0 \\ 1 & 0 & 0 & r-2 & 0 \\ 1 & 0 & 0 & 0 & -2 \end{pmatrix}$$

which bounds  $\Sigma(K)$  when  $r = -1$ .

Setting  $r = +1$ , we obtain a four-manifold which bounds the branched double-cover of the  $(2, 5)$  torus knot. This can be seen by replacing the  $-3/2$  twist inside the oval marked by  $x$  in the picture for  $10_{151}$  in Figure 7 by a single  $-1/2$  twist. A direct calculation shows that this four-manifold is sharp. Moreover, the  $4 \times 4$  submatrix obtained by deleting the last row and column is a Goeritz matrix for the alternating knot with determinant 19 obtained by forming the  $K_1$ -resolution of any of the three crossings in the oval marked by  $x$ . In particular, the associated four-manifold there is

$$\begin{array}{ccccccc} & -1 & 0 & -r & 0 & -1 & \end{array}$$

FIGURE 8. **A chain of unknots.**

sharp, as well. Since the determinant of  $G$  (for arbitrary  $r$ ) is given by  $-24 + 19r$ , it follows from Proposition 2.6 (and a descending induction starting at  $r = +1$ ) that the four-manifold described by our plumbing of unknots with framing  $r \leq 1$  is sharp.

Thus, we can calculate the correction terms for  $\Sigma(10_{151})$  using  $G$  at  $r = -1$  to get  $A$ . One then checks that the only even, positive matching is given by

$$10_{151} : \quad 2, 2, 2, 4, 4, \mathbf{6}, 4, 4, 2, 2, 2, 2.$$

We proceed similarly for  $K = 10_{158}$ . Again, we find a chain is the plumbing description for  $10_{158}$  as given in Figure 8, which we replace by an unknot with framing  $r - 2$ . Thus, we obtain a four-manifold  $X_K$  which bounds  $\Sigma(K)$ , with intersection form given by

$$G = \begin{pmatrix} -4 & 2 & 2 & 1 \\ 2 & -4 & 1 & 0 \\ 2 & 1 & -4 & 0 \\ 1 & 0 & 0 & r - 2 \end{pmatrix}$$

with  $r = -1$ . Ignoring the unknot with framing  $r - 2$ , we obtain the Goeritz matrix for the alternating link obtained from the  $K_1$ -resolution. When  $r = +1$ , we obtain a still negative-definite intersection form for a four-manifold which bounds  $\Sigma(4_1)$ . Again, a direct calculation shows that this four-manifold is sharp, and hence, by Proposition 2.6, so is  $X_K$  (indeed since  $G$  has matrix  $25 - 20r$ , the corresponding four-manifold is sharp for all  $r \leq 1$ ). Calculating the vector  $A$  using the matrix  $G$ , we see that there is a unique even, positive matching:

$$10_{158} : \quad 2, 2, 2, 2, 4, 4, \mathbf{6}, 4, 4, 2, 2, 2.$$

Next, we turn to the knot  $K = 10_{162}$  (according to a numbering scheme in which  $10_{161} \neq 10_{162}$ ). Like in  $10_{148}$ , when constructing  $X_K$ , we can blow down a  $+1$ -framed unknot to obtain a four-manifold  $X_K$  with intersection form given by

$$G = \begin{pmatrix} -5 & 2 & 2 & 0 \\ 2 & -5 & 2 & 0 \\ 2 & 2 & -4 & 1 \\ 0 & 0 & 1 & r - 1 \end{pmatrix}$$

when  $r = -1$ . The  $3 \times 3$  submatrix on the upper left corner specifies an intersection form for the manifold which bounds  $\Sigma(K_1)$ , while setting  $r = 0$  specifies an intersection form for a four-manifold which bounds  $\Sigma(K_0)$ . It is easy to see that both  $K_1$  and  $K_0$  have alternating projections –  $K_1$  is an 8-crossing link with determinant 28, while  $K_0$  is the knot  $5_2$  (with determinant 7). Comparing against the Goeritz matrix for the alternating projection, we see that the four-manifold specified by  $G$  with  $r = 0$  is sharp (for  $\Sigma(K_0)$ ). Thus, by Proposition 2.6, the four-manifold  $X_K$  is sharp, as well and hence the correction terms for  $\Sigma(10_{162})$  can be calculated using the matrix  $G$  with  $r = -1$ .

The unique even, positive matching is given now by

$$10_{162} : \quad 2, 2, 4, \mathbf{4}, 4, 2, 2, 2.$$

Note that none of the even, symmetric matchings for  $10_{148}$ ,  $10_{151}$ ,  $10_{158}$ , and  $10_{162}$  listed above is symmetric, and hence none of these knots has unknotting number equal to one, by Theorem 4.1.

**7.3. Final remarks.** In fact, the methods described here – cyclicity of  $H_1(\Sigma(K); \mathbb{Z})$ , and the methods from Theorem 1.1 – are sufficient to classify all 10-crossing knots with  $u = 1$ , with the two exceptions  $10_{145}$  and  $10_{153}$ . In particular, the knots  $10_{140}$ ,  $10_{163}$ ,  $10_{165}$  admit no even matching, while  $10_{144}$  admits no even, positive matching. (Note that  $10_{131}$  is listed as having unknown unknotting number in some sources, but it can be unknotted in one step according to Figure 9, by changing the indicated crossing, see also [34].)

The knot  $10_{145}$  has  $u = 2$  according to [35], while  $10_{153}$  has  $u = 2$  according to [6], completing the classification of all knots with  $u = 1$  and  $\leq 10$  crossings.

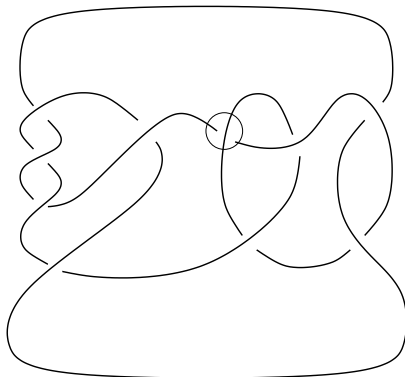


FIGURE 9. **Unknotting**  $10_{131}$ .



## 8. REFINEMENTS

**8.1. Signed unknottings.** The reason for the two choices of  $\epsilon$  in the statements of Theorems 1.1 rests on orientations. On the one hand, the condition on a knot of having unknotting number one does not distinguish a knot  $K$  from its mirror  $r(K)$ . By contrast, the branched double-cover of  $S^3$  with its standard orientation along a knot  $K$  endows  $\Sigma(K)$  with a natural orientation, with the property that  $\Sigma(r(K)) \cong -\Sigma(K)$ ; moreover the signs of the correction terms depend on a choice of orientation for its underlying three-manifold.

A refined statement of this result can be formulated which makes use of a more orientation-dependent notion of unknotting number one: we could consider knots  $K$  which can be unknotted by changing a single negative crossing to a positive crossing for some projection  $K$  (we use here the usual conventions from knot theory as illustrated in Figure 10). One obstruction to this sign-refined question is the signature  $\sigma(K)$  of the knot. Specifically, recall that the signature satisfies the inequality

$$(32) \quad \sigma(K_-) - 2 \leq \sigma(K_+) \leq \sigma(K_-),$$

and hence a knot with this property has  $\sigma = 0$  or  $2$ .

For the purpose of the following statement, recall that we use the coloring convention illustrated in Figure 1. Given an alternating knot with determinant  $D$ , fix a regular alternating projection and fix a corresponding Goeritz matrix  $Q$ , using the white graph as described in the introduction. Given an isomorphism  $\phi: \mathbb{Z}/D\mathbb{Z} \rightarrow \text{Coker}(q)$ , let

$$T_\phi(i) = (-1)^{\frac{\sigma(K)}{2}} \cdot M_Q(\phi(i)) - \gamma_D(i).$$

**Theorem 8.1.** *Let  $K$  be an alternating knot which can be unknotted by changing a single negative crossing to a positive one in some (not necessarily alternating) projection of  $K$ , then there is an isomorphism  $\phi: \mathbb{Z}/D\mathbb{Z} \rightarrow \text{Coker}(q)$  with the properties that for all  $i \in \mathbb{Z}/D\mathbb{Z}$ :*

$$\begin{aligned} T_\phi(i) &\equiv 0 \pmod{2} \\ T_\phi(i) &\geq 0. \end{aligned}$$

*If in addition  $|M_{Q_K}(0)| \leq \frac{1}{2}$ , then there is a choice of  $\phi$  satisfying the above to constraints, and the following additional symmetry:*

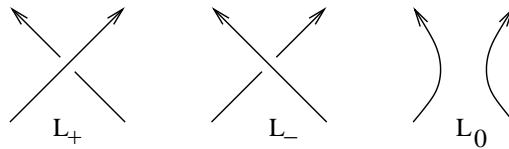


FIGURE 10. **Sign conventions on crossings.** Crossings of type  $L_+$  are positive,  $L_-$  is negative, and  $L_0$  is the oriented resolution.

$$T_\phi(i) = T_\phi(2k - i)$$

for  $1 \leq i < k$  when  $D = 4k - 1$  and for  $0 \leq i < k$  when  $D = 4k + 1$ .

**Proof.** We need the following precise version of Montesinos' lemma, see [34]: if  $K$  is a knot with determinant  $D = 2n + 1$  which can be unknotted by changing a negative crossing to a positive one, then  $\Sigma(K) = S^3_{-\epsilon, \frac{2n+1}{2}}(C)$  where  $\epsilon = (-1)^{\frac{\sigma}{2}}$ . (Note that if  $K$  can be unknotted by changing a negative crossing to a positive one, then  $\sigma(K) = 0$  or  $2$  according to Inequality (32).)

We have the two triples of three-manifolds  $(\Sigma(K_0), \Sigma(K_+), \Sigma(K_1))$  and  $(\Sigma(K_1), \Sigma(K_-), \Sigma(K_0))$  which are related by two-handle additions as in the hypothesis of Theorem 2.2. Let  $A$  denote the two-handle cobordism from  $\Sigma(K_1)$  to  $\Sigma(K_-)$  and  $B$  denote the two-handle from  $\Sigma(K_+)$  to  $\Sigma(K_1)$ . Note that here  $K_0$  is the oriented resolution of  $K_\pm$ . By handlesliding, it is easy to see that  $A \circ B$  contains a sphere with square  $-2$ , and another linking two-handle. Our assumption that  $K_+$  is the unknot then ensures that  $\Sigma(K_-)$  can be written as  $S^3_{\pm \frac{2n+1}{2}}$ , where the sign depends on  $b_2^+(A \circ B)$ .

It is a standard fact that for a knot  $K$   $(-1)^{\frac{\sigma(K)}{2}} D = \Delta_K(-1)$  (c.f. [13]). Suppose that  $\sigma(K)$  is zero. Then the above fact, together with the skein relation for the Alexander polynomial gives that  $\det(K_0) = n$ . It follows that  $\det(K_1) = n + 1$ . These in turn ensure that both cobordisms  $A$  and  $B$  are negative-definite, and hence that  $\Sigma(K) = \Sigma(K_-) = S^3_{-\frac{2n+1}{2}}(C)$ .

In the case where  $\sigma(K) = 2$ , the same argument now proves that  $\det(K_0) = n + 1$  and  $\det(K_1) = n$ , from which it follows that  $b_2^+(A) = 1$ . Thus, the Kirby calculus picture for  $A \circ B$  consists of a knot  $C$  with positive framing, with a linking unknot with framing  $-2$ . By modifying the cobordism  $A \circ B$  in a straightforward way, we can trade the linking unknot for another one with framing  $+2$ , at the cost of increasing the framing on  $C$  by one. Thus, we have expressed  $\Sigma(K) = \Sigma(K_-) \cong S^3_{\frac{2n+1}{2}}(C)$ .

With the signs pinned down, now, the proof of the result follows from the proof of Theorem 1.1.  $\square$

More informally, if an alternating knot has unknotting number equal to one and  $|M_{Q_K}(0)| \leq \frac{1}{2}$ , but it has positive, even, symmetric matchings with only one choice of  $\epsilon$ , then it does not have two one-step unknottings with different signs.

Some knots – for example,  $8_{13}$  – can be unknotted by single crossing-changes with either sign. According to Inequality (32), a knot with this property must have vanishing signature. Thus, we illustrate Theorem 8.1 with a knot whose signature vanishes.

Consider the knot  $9_{33}$  illustrated in Figure 11. For one choice of  $\epsilon$ , there is a single even, positive matching of the form

$$2, 2, 2, 2, 4, 4, 4, 6, 6, 8, 6, 6, 4, 4, 2, 2, 2, 2,$$

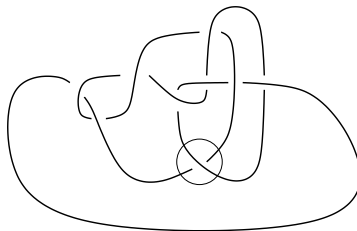


FIGURE 11. **The knot  $9_{33}$ .** By changing the circled indicated (negative) crossing, we can unknot  $9_{33}$ . However, no projection of  $K = 9_{33}$  has a positive crossing which, when changed, unknots  $K$ .

which is evidently asymmetric, while for the opposite choice of  $\epsilon$ , we have the following unique even, positive matching which is symmetric:

$$(33) \quad 2, 2, 2, 2, 4, 4, 4, 6, 6, 8, 6, 6, 4, 4, 4, 2, 2, 2, 2$$

(the existence of at least one such a matching is guaranteed from Theorem 1.1, as  $9_{33}$  has unknotting number equal to one). This shows, however, that there is no one-step unknotting which involves a crossing change with the opposite sign.

**8.2. Interpreting the matchings.** If  $K$  is an alternating knot with unknotting number equal to one and  $|M_{Q_K}(0)| \leq \frac{1}{2}$ , then the even integers  $T_{\phi, \epsilon}$  guaranteed by Theorem 1.1 have a concrete topological interpretation, which we now explain.

If  $K$  is a knot with unknotting number one, then of course we can draw an arc  $\gamma$  in  $S^3$  which connects two points of  $K$ , with the property that a standard modification of  $K$  in a tubular neighborhood of  $\gamma$  gives us the unknot. From a dual point of view, a knot with unknotting number equal to one can be specified by an unknot together with a (framed) arc  $\delta$  (in a neighborhood of which we modify the unknot to get  $K$ ). We call  $\delta$  a *knotted arc* for  $K$ .

On the other hand, if  $\delta$  is an arc connecting two points on an unknot  $O$ , we can construct a knot  $C$  in  $S^3$ , thought of as the branched double-cover of  $O$ . We claim that if  $K$  is a knot which satisfies all the hypotheses of Theorem 1.1, then the (even) integers appearing as differences between correction specify the Alexander polynomial of  $C$  (up to some finite indeterminacy determined by the possible choices of  $\epsilon$  and  $\phi$ ).

To state the result, it is convenient to reformulate the information in the Alexander polynomial of  $C$ . Let  $C$  be a knot in  $S^3$ , and write its symmetrized Alexander polynomial as

$$\Delta_C(T) = a_0 + \sum_{i>0} a_i(T^i + T^{-i}).$$

then, its torsion coefficients  $t_i(C)$  are given by the formula:

$$t_i(C) = \sum_{j=1}^{\infty} j \cdot a_{|i|+j}.$$

It will also be convenient to have the following notation. If  $C \subset S^3$  is a knot, then an orientation for  $C$  specifies a map

$$\sigma: \mathbb{Z}/p\mathbb{Z} \longrightarrow S_p^3(C)$$

by the condition that  $\sigma(i)$  extends over the two-handle cobordism  $W_p(C)$  from  $S^3$  to  $S_p^3(C)$  as a  $\text{Spin}^c$  structure  $\mathfrak{s}$  with

$$\langle c_1(\mathfrak{s}), [\widehat{C}] \rangle \equiv 2i - p \pmod{2p}.$$

**Theorem 8.2.** *Let  $C \subset S^3$  be a knot in  $S^3$  with the property that  $S_p^3(C)$  is an  $L$ -space  $Y$ , then*

$$2t_i(C) = \begin{cases} -d(S_p^3(C), \sigma(i)) + d(S_p^3(O), \sigma(i)) & \text{if } 2|i| \leq p \\ 0 & \text{otherwise,} \end{cases}$$

The above is essentially a restatement of Corollary 7.5 of [22], only that result is stated in the case where  $Y$  is the lens space  $L(p, q)$ ; however, the only property about lens spaces used in its proof is that lens spaces are  $L$ -spaces. (In fact, the result is seen as a consequence of a stronger result, which describes  $HF^+(S_0^3(C))$  in terms of the correction terms for  $S_p^3(C)$  and the map  $\sigma$ .)

Thus, in view of Theorem 4.2, if  $K$  is an alternating knot with unknotting number equal to one, then twice the torsion coefficients of the branched double-cover of the knotting arc must appear (in order) in some matching for  $K$ . Incidentally, from this point of view, the symmetric condition on the matching corresponds to the usual symmetry of the Alexander polynomial of  $C$ .

For example, we saw earlier that the knot  $9_{33}$  has a unique even, positive, and symmetric matching, as given in Equation (33). Indeed, converting from the torsion back to the Alexander polynomial, it follows that the Alexander polynomial of  $C$  is given by

$$-1 + (T^2 + T^{-2}) - (T^4 + T^{-4}) + (T^5 + T^{-5}) - (T^8 + T^{-8}) + (T^9 + T^{-9})$$

which, incidentally, is the Alexander polynomial of the  $(7, 4)$  torus knot. It is, in fact, reasonable to expect that if  $\delta$  is any knotting arc of  $9_{33}$ , then its branched double-cover is the  $(7, 4)$  torus knot. (Compare Berge's conjecture on knots which admit lens space surgeries, c.f. [1], [9].)

### 8.3. Alternating knots with unknotting number one as a source of examples.

Alternating knots with unknotting number equal to one can be viewed as a wide source of knots in  $S^3$  which admit  $L$ -space surgeries (by taking the branched double-cover of the knotting arc). Such knots are very special. For example, in [20], we prove the following:

**Theorem 8.3.** *Suppose that  $C$  is a knot in  $S^3$  which admits an integral  $L$ -space surgery, then all the coefficients of its Alexander polynomial are  $\pm 1$  and the non-zero coefficients all alternate in sign.*

The above theorem appears as Corollary 1.3 in [20], where it is seen as a corollary to a more general result which constrains the structure of the “knot Floer homology” of  $K$ , c.f. [21] and [30]. Indeed combining results from [20], [26] and [23], we get that for such a knot, the Seifert genus, the four-ball genus, and the degree of the Alexander polynomial all agree. Moreover, hyperbolic knots with  $L$ -space surgeries all provide infinitely many examples of hyperbolic three-manifolds which admit no taut foliation [26]. (Compare also [12] for analogous results in the realm of Seiberg-Witten monopole Floer homology.)

Berge’s construction [1] gives many examples knots with  $L$ -space surgeries. Alternating knots with unknotting number equal to one provide another source of such examples.

**8.4. Stronger forms of Theorem 1.1 using  $L$ -space surgeries theorems.** Results from [20] stated above concerning the structure of the Alexander polynomial of a knot admitting  $L$ -space surgeries can be viewed as giving further restrictions on the positive, even, symmetric matchings associated to alternating knots with unknotting number equal to one. For example, we obtain the following result:

**Theorem 8.4.** *If  $K$  is an alternating knot with unknotting number equal to one with determinant  $D = 4k \pm 1$ , then there is a choice of isomorphism  $\phi: \mathbb{Z}/D\mathbb{Z} \rightarrow \text{Coker } q$  and  $\epsilon$ , with the property that the matching  $T_{\phi, \epsilon}$  satisfies the following restrictions:*

- $T_{\phi, \epsilon}(i) \equiv 0 \pmod{2}$ ,
  - $T_{\phi, \epsilon}(i) \geq 0$  for all  $i$ ; moreover for  $i = 1, \dots, k-1$
- (34)  $T_{\phi, \epsilon}(i) \leq T_{\phi, \epsilon}(i+1) \leq T_{\phi, \epsilon}(i) + 2$
- $T_{\phi, \epsilon}(i) = T_{\phi, \epsilon}(2k-i)$  for  $1 \leq i < k$ . If, in addition,  $|M_{Q_K}(0)| \leq \frac{1}{2}$ , the above symmetry extends to  $i = 0$  when  $D = 4k + 1$ .

The proof is based on a combination of Theorem 8.3, together with the techniques of this paper. Before giving the proof, we give the following improvement of Theorem 4.2 (which we state in the notation from Section 4):

**Theorem 8.5.** *Assume that  $C \subset S^3$  is a knot with the property that for some  $n > 1$   $S^3_{-(2n-1)/2}(C)$  is an  $L$ -space. Then so is  $S^3_{-n}(C)$ , and also for  $i = 1, \dots, 2n-2$ ,*

$$d(S^3_{-n}(C), v_i) - d(S^3_{-n}(O), v_i) = d(S^3_{-(2n-1)/2}(C), w_i) - d(S^3_{-(2n-1)/2}(O), w_i).$$

**Proof.** We proceed as in the proof of Theorem 4.2, only we no longer have a hypothesis which ensures that  $d(S^3_{-n}(C), v_j) - d(S^3_{-n}(O), v_j) = 0$  for  $v_j = v_0$  or  $v_{2k}$  (depending on the parity of  $n$  as in that proof); we have only that  $0 \leq d(S^3_{-n}(C), v_j) - d(S^3_{-n}(O), v_j)$ . However, by Theorem 8.3, it follows that  $d(S^3_{-n}(C), v_j) - d(S^3_{-n}(O), v_j) \leq 2$  (since this difference is twice some coefficient of the Alexander polynomial of  $C$ ). It follows now that from this, together with the argument from the proof of Theorem 4.2 that the inequality

$$d(S^3_{-n}(C), v_i) - d(S^3_{-n}(O), v_i) \leq d(S^3_{-\frac{2n-1}{2}}(C), w_i) - d(S^3_{-\frac{2n-1}{2}}(O), w_i)$$

(from Lemma 4.3) can be a strict inequality at most one value of  $i = 0, \dots, 2n - 2$  (note that  $d(S_{-n}^3(C), v_i) - d(S_{-n}^3(O), v_i) \equiv 0 \pmod{2}$ ). By the symmetry of the  $w_i$  sending  $i \mapsto 2n - 1 - i$ , it follows at once that the inequality can be strict only when  $i = 0$ .  $\square$

**Proof of Theorem 8.4.** Proceeding as in the earlier proof, and applying Theorem 8.5 if necessary, we see that for  $0 \leq i < k$  the numbers  $T_{\phi, \epsilon}(k - i)$  are differences in correction terms for  $S_{-n}^3(C)$  and  $S_{-n}^3(O)$ . In turn, according to Theorem 8.2, these are identified with torsion coefficients for  $C \subset S^3$ . Now, Equation (34) is a consequence of this fact, together with Theorem 8.3.  $\square$

As we saw, Theorem 1.1 suffices for the study of knots with  $\leq 10$ , but it is possible that for other applications, the stronger form given in Theorem 8.4 might be useful.

## REFERENCES

- [1] J. O. Berge. Some knots with surgeries giving lens spaces. Unpublished manuscript.
- [2] S. A. Bleiler. Knots prime on many strings. *Trans. Amer. Math. Soc.*, 282(1):385–401, 1984.
- [3] G. Burde and H. Zieschang. *Knots*. Number 5 in de Gruyter Studies in Mathematics. Walter de Gruyter & Co., 1985.
- [4] M. Culler, C. McA. Gordon, J. Luecke, and P. B. Shalen. Dehn surgery on knots. *Ann. of Math*, 125(2):237–300, 1987.
- [5] K. A. Frøyshov. The Seiberg-Witten equations and four-manifolds with boundary. *Math. Res. Lett*, 3:373–390, 1996.
- [6] C. McA. Gordon and J. Luecke. Knots with unknotting number one and essential conway spheres. In preparation.
- [7] T. Kanenobu and H. Murakami. Two-bridge knots with unknotting number one. *Proc. Amer. Math. Soc.*, 98(3):499–502, 1986.
- [8] A. Kawauchi. *A survey of knot theory*. Birkhäuser Verlag, Basel, 1996.
- [9] R. Kirby. Problems in low-dimensional topology. In *Geometric Topology*, volume 2.2 of *AMS/IP Stud. Adv. Math.*, pages 35–473. Amer. Math. Soc., 1997.
- [10] T. Kobayashi. Minimal genus Seifert surfaces for unknotting number 1 knots. *Kobe J. Math.*, 6(1):53–62, 1989.
- [11] P. B. Kronheimer and T. S. Mrowka. Gauge theory for embedded surfaces. I. *Topology*, 32(4):773–826, 1993.
- [12] P. B. Kronheimer, T. S. Mrowka, P. S. Ozsváth, and Z. Szabó. Monopoles and lens space surgeries. math.GT/0310164, to appear in *Ann. Math.*
- [13] W. B. R. Lickorish. The unknotting number of a classical knot. In *Combinatorial methods in topology and algebraic geometry (Rochester, N.Y., 1982)*, volume 44 of *Contemp. Math.*, pages 117–121. Amer. Math. Soc., Providence, RI, 1985.
- [14] J. M. Montesinos. Surgery on links and double branched covers of  $S^3$ . In *Knots, groups, and 3-manifolds (Papers dedicated to the memory of R. H. Fox)*, pages 227–259. Ann. of Math. Studies, No. 84. Princeton Univ. Press, Princeton, N.J., 1975.
- [15] H. Murakami and A. Yasuhara. Four-genus and four-dimensional clasp number of a knot. *Proc. Amer. Math. Soc.*, 128(12):3693–3699, 2000.
- [16] K. Murasugi. On a certain numerical invariant of link types. *Trans. Amer. Math. Soc.*, 117:387–422, 1965.
- [17] Y. Nakanishi. A note on unknotting number. *Math. Sem. Notes Kobe Univ.*, 9(1):99–108, 1981.
- [18] B. Owens and S. Strle. Rational homology spheres and four-ball genus. math.GT/0308073, 2003.
- [19] P. S. Ozsváth and Z. Szabó. Holomorphic triangles and invariants for smooth four-manifolds. math.SG/0110169.
- [20] P. S. Ozsváth and Z. Szabó. On knot Floer homology and lens space surgeries. math.GT/0303017.
- [21] P. S. Ozsváth and Z. Szabó. Holomorphic disks and knot invariants. math.GT/0209056, 2002.
- [22] P. S. Ozsváth and Z. Szabó. Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary. *Adv. Math.*, 173(2):179–261, 2003.
- [23] P. S. Ozsváth and Z. Szabó. Knot Floer homology and the four-ball genus. *Geometry and Topology*, 7:615–643, 2003.
- [24] P. S. Ozsváth and Z. Szabó. On the Floer homology of plumbed three-manifolds. *Geometry and Topology*, 7:185–224, 2003.
- [25] P. S. Ozsváth and Z. Szabó. On the Heegaard Floer homology of branched double-covers. math.GT/0309170, 2003.

- [26] P. S. Ozsváth and Z. Szabó. Holomorphic disks and genus bounds. *Geom. Topol.*, 8:311–334 (electronic), 2004.
- [27] P. S. Ozsváth and Z. Szabó. Holomorphic disks and three-manifold invariants: properties and applications. *Ann. Math.*, 159(3):1159–1245, 2004.
- [28] P. S. Ozsváth and Z. Szabó. Holomorphic disks and topological invariants for closed three-manifolds. *Ann. Math.*, 159(3):1027–1158, 2004.
- [29] P. S. Ozsváth and Z. Szabó. Knot Floer homology, genus bounds, and mutation. *Topol. Appl.*, 141(1-3):59–85, 2004.
- [30] J. Rasmussen. *Floer homology and knot complements*. PhD thesis, Harvard University, 2003.
- [31] Dale Rolfsen. *Knots and links*, volume 7 of *Mathematics Lecture Series*. Publish or Perish Inc., Houston, TX, 1990. Corrected reprint of the 1976 original.
- [32] L. Rudolph. Quasipositivity as an obstruction to sliceness. *Bull. Amer. Math. Soc. (N.S.)*, 29(1):51–59, 1993.
- [33] M. G. Scharlemann. Unknotting number one knots are prime. *Invent. Math.*, 82(1):37–55, 1985.
- [34] A. Stoimenow. Polynomial values, the linking form and unknotting numbers. *Math. Res. Lett.*, 11(5-6):755–769, 2004.
- [35] T. Tanaka. Unknotting numbers of quasipositive knots. *Topology Appl.*, 88(3):239–246, 1998.

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